Numerical steepest descent — "pain-free numerical integration of osc. analytic funcs." — Barnett 10/2/18

warm-up = quadrature, "inf." trap rule. — e.g. — apps: · coral. Green's funcs, waves · collab w/ Jason Kaye, Leslie Greengard

$f: \mathbb{R} \to \mathbb{C}$ $\quad \int_{-\infty}^{\infty} f(x)\,dx \approx h \sum_{j \in \mathbb{Z}} f(x_j) \qquad x_j = hj$.

How error scales w/ h? $\qquad \| \qquad \qquad \|$ Poiss. sum formula

$$\hat f(0) \qquad \qquad \sum_{m \in \mathbb{Z}} \hat f\left(\frac{2\pi}{h} m\right)$$

$$\sum_l \psi(x - lh) = \frac{1}{h} \sum_m \hat\psi \ldots$$

$$\text{ie } h \sum_j f(hj) = \sum_m \hat f\left(\frac{2\pi}{h} \ldots\right)$$

take diff to get error.

so error $E_h[f] = \left| \sum_{m \neq 0} \hat f\left(\frac{2\pi}{h} m\right) \right|$ where $\hat f(k) := \int f(x) e^{ikx} dx$

So if f bandlimited out to almost twice Nyquist, exact! ($k_{Nyq} = \pi/h$)

$\hat f$: $-2\pi/h \quad 0 \quad 2\pi/h \quad 4\pi/h \quad k$

A) gaussian $f(x) = e^{-x^2/2} \quad \leftrightarrow \quad \hat f(k) = \sqrt{2\pi}\, e^{-k^2/2}$

dominate. $\quad E_h \approx \sqrt{2\pi} \cdot 2 e^{-\frac{1}{2}\left(\frac{2\pi}{h}\right)^2} \approx 5 e^{-20/h^2}$ abs.

(plot it & compare!) $\quad h = 0.8$ enough for 10^{-13} rel. err., but $\quad f(8) \approx 10^{-14}$.

Show $e^{-z^2/2}$ in $\mathbb{C}$ $\quad$ ~~big~~ $\quad$ so only need pts in $[-8,8]$, ≈ 20 of them ie $j = -10, -9, \ldots 10$.

B) Osc. funcs. $g(x) = \frac{1}{1+x^2} \cos(30(x-1)^2)$ — plot: → notice its ... check ✓.

seek $I = \int_{-\infty}^{\infty} g(x)\,dx$. slow ampl. func. — large — phase func. — real part of

$$f(z) = \frac{1}{1+z^2} e^{i\,30(z-1)^2} \quad \text{form } a(x)\, e^{i\, b(x)}$$

analytic: plot in $\mathbb{C}$. $\quad$ note CTF in cryo is similar form, radially.

stat. phase pt

5in $\quad x_0 = 1$: where $b'(x_0) = 0$

locally like gaussian rot. by $+\pi/4$. : what scale? $\quad b''(x_0) = 60 \quad$ cf $\left(\frac{x^2}{2}\right)'' = 1$ for std. Gaussian

so zoom in by $\sqrt{60}$

so choose $h \leftarrow \frac{e^{+i\pi/4}}{\sqrt{60}} h$

chk conv ✓ $\quad$ 1e-12 $\quad$ w in code

Closing: can do multi stat phase pts, change var to handle → ∞. curved (analytic!) contours. or Gauss-Legendre on straight segments.

Show JCP.