Fresnel Diffraction for Starshade Design using the Non-Uniform FFT

Alex Barnett[1]

SIAM Annual Meeting, 7/21/21

Thanks to: David Hogg, Leslie Greengard, David Spergel, for introducing me to the problem
Stuart Shaklan, Anthony Harness, Philip Dumont, for discussions

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What on Earth is a starshade? (PS: it's not!)

"Exoplanet" = planet* orbiting some distant sun $\quad$ * that might have life!

Can't image an exoplanet directly because it's sun is $10^{10}\times$ brighter and only $\sim 10^{-7}$ radians separated in apparent angle: detector dazzled!

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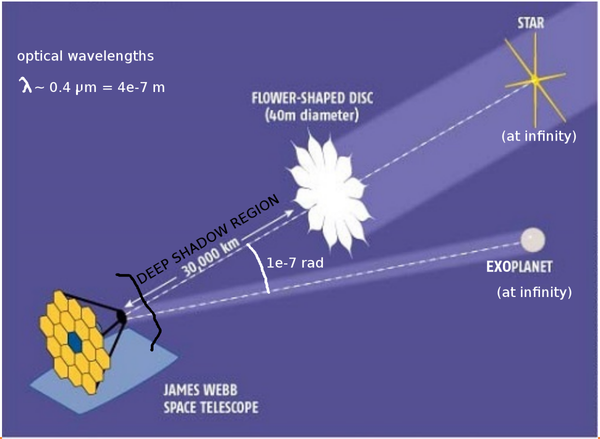
Thus, plan is: block *only the starlight* via "occulter" floating in space...



- a lovely shape optimization problem: pointy "petals" minimize starlight diffraction into shadow (kills Poisson spot)

  (Vanderbei et al '07)

- Big $ NASA/JPL project: video

Crash course in diffraction approximations

The occulter is basically a lump R of conductive metal floating in $\mathbb{R}^3$

Light obeys a PDE: Maxwell's equations relating six field components $(\boldsymbol{E}, \boldsymbol{B})$, with approx. conductor boundary conditions (transverse $\boldsymbol{E} = \mathbf{0}$)

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Scalar approximation to Maxwell applies for length-scales much larger than the wavelength λ, would give acoustic scattering BVP:

$$(\Delta + (2\pi/\lambda)^2)u = 0 \qquad \text{in } \mathbb{R}^3 \backslash \overline{R}$$

Helmholtz PDE Δ = Laplacian

$$u = -u_{\text{incident}} \qquad \text{on } \partial R$$

$\leftarrow$ surface of R $+$ radiation cond. at ∞

occulter $10^7\lambda$ across: $\gg 10^{14}$ unknowns even using boundary integral equations :(

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But R is a thin sheet lying in a plane, so *Kirchhoff approximation* gives direct solution as Green's integral (single-layer potential w/ unit density) over the transmitting part $\Omega \subset \mathbb{R}^2$ of this "source plane":

$$u_{\text{target}} \approx \iint_\Omega \frac{e^{-2\pi i \rho/\lambda}}{\rho}\, dx dy$$

$\rho =$ distance from source plane point to target

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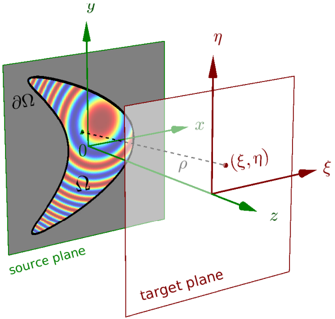
Finally, *Fresnel approx.* truncates Taylor: $\quad \rho = \sqrt{z^2 + r^2} \approx z + \frac{r^2}{2z}$

Pythagoras: $z =$ downstream distance ($\sim$ 3e7 m), $r =$ transverse distance ($\sim$ 10 m)

Fresnel scalar diffraction setup and task

Let $\Omega \subset \mathbb{R}^2$ be occulter (eg, starshade) in plane $z = 0$

Incident plane wave $e^{2\pi i z/\lambda}$ along z-axis: seek field u^{oc} in target plane



Write as *aperture* problem: (Babinet ∼1830)

$$u^{\text{oc}}(\xi,\eta) \;=\; 1 - u^{\text{ap}}(\xi,\eta)\ ,$$

$$u^{\text{ap}}(\xi,\eta) = \frac{1}{i\lambda z}\iint_\Omega e^{\frac{i\pi}{\lambda z}\left[(\xi-x)^2+(\eta-y)^2\right]}\,dxdy$$

$\lambda =$ wavelength $\quad z =$ downstream dist.

- characteristic func. of aperture χ_Ω convolved in 2D w/ complex Gaussian

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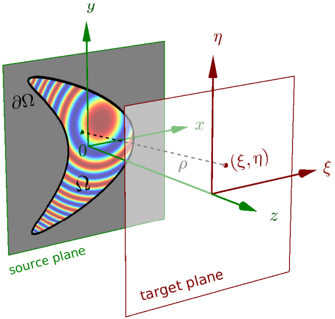

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Fresnel number $\mathfrak{f} := \dfrac{R^2}{\lambda z} \sim 5$ to 20 for starshades $\qquad R =$ aperture radius

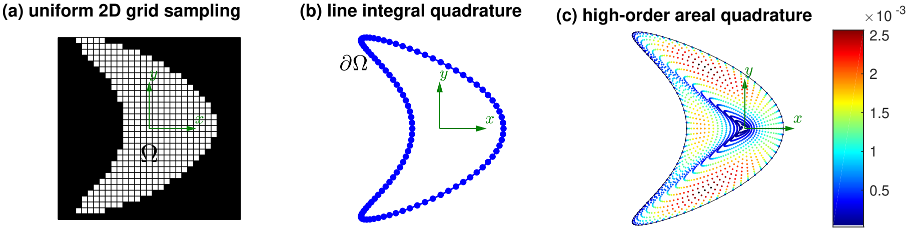
Is scalar approx. good? yes $\qquad$ for full scale; not perfect for scale models

Is Fresnel approx. good? yes! next term $\frac{R^4}{\lambda z^3} \sim 10^{-7}$ $\qquad$ even for scale models

- Numerical tolerance? u^{ap} abs error $< 10^{-6}$ $\qquad$ to model shadow intensity 10^{-10}

Numerical methods for Fresnel diffraction

Seek u^{ap} on, say, $n \times n$ grid. Two existing methods; we propose a third...



2D FFT (or pair) convolution

fast $\mathcal{O}(n^2 \log n)$

(Mas, Lo, Junchang et al)

low-order $\mathcal{O}(1/n)$

gray-pixel at best $\mathcal{O}(1/n^2)$

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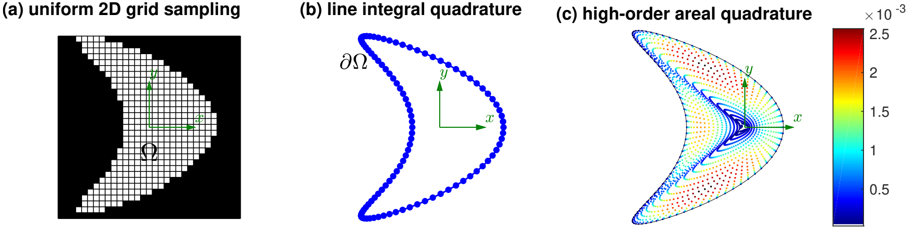

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direct summation

$u^{\text{ap}} = \frac{1}{2\pi} \int_{\partial\Omega} \left(1 - e^{\frac{i\pi}{\lambda z} r^2}\right) \frac{\mathbf{r} \times d\mathbf{s}}{r^2}$

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(Miyamoto–Wolf, Dauger, Cash, Cady, Barnett '21)

high-order accurate

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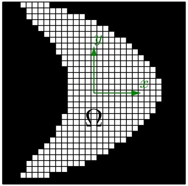
(a) uniform 2D grid sampling



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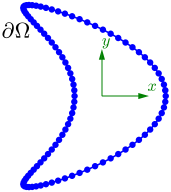
(b) line integral quadrature



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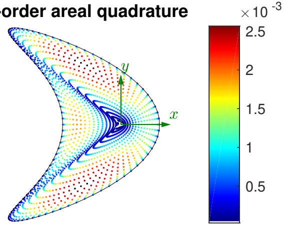
(c) high-order areal quadrature



2D nonuniform FFT

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high-order accurate

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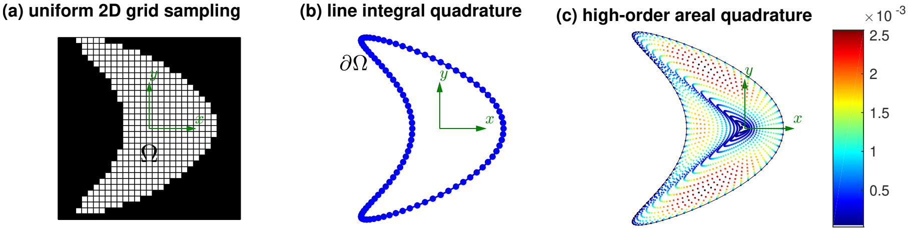

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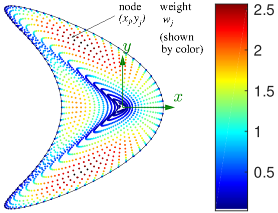
high-order accurate

- prior starshade design/modeling used slow method (b), since (a) inacc.
- JPL code (BDWF) (Cady '12) also blows up for targets near $\partial\Omega$

We need areal quadrature (AQ) over aperture Ω

AQ is simply set of nodes (x_j, y_j), $j = 1, \ldots, N$, with weights w_j, so



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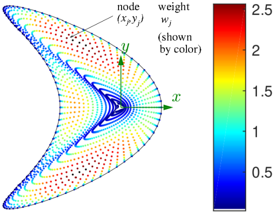

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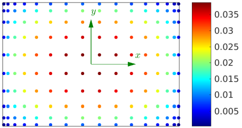
Eg: Ω = rectangle:

tensor product Gauss–Legendre rule



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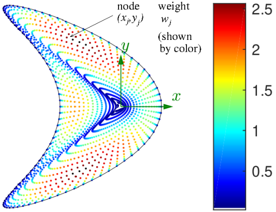

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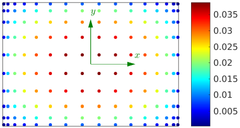

AQ for "arbitrary" geometries?

- easy from existing line (edge) integral quadrature rule, via *dilation*
- unions of simple shapes + smooth transformations — Jacobean scales w_j
- auto-generate from CAD format (not yet)

Issues: geometry formats, precise ($< 10^{-5}$) tolerances

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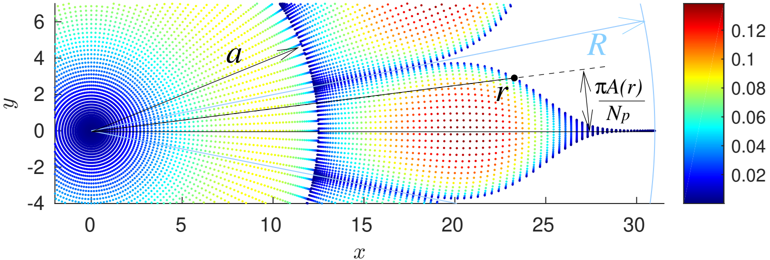
Areal quadrature for starshades

Easy to build AQ for "ideal" starshade $N \sim 10^4$–10^6, err 10^{-6}, 20 lines MATLAB

High-order interpolation from nodes giving petal apodization func $A(r)$:



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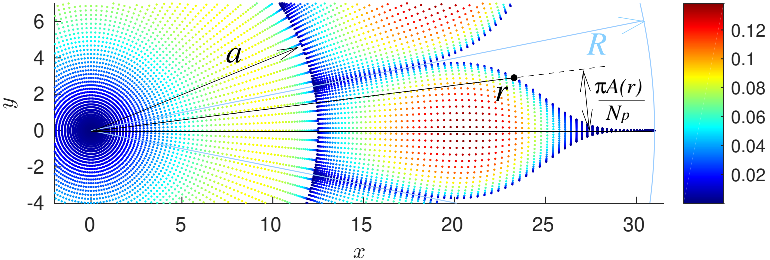

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Real starshade: defects, roughness, misalignments, thermal distortion...

- can add/subtract a variable-width strip region to an ideal one
- statistical shape sampling: many forward simulations, estim. p(failure)

Communicating *precise* geometries w/ engineers can be hard!

The fast trick: factorization of quadratic exponential

For all targets $k = 1, \ldots, M$, eval. quadrature rule for Frensel integral:

$$u_k^{\text{ap}} \approx \frac{1}{i\lambda z} \sum_{j=1}^{N} e^{\frac{i\pi}{\lambda z}\left[(\xi_k - x_j)^2 + (\eta_k - y_j)^2\right]} w_j$$

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$\nearrow$ iii) post-multiply $\qquad$ $\uparrow$ ii) 2D "type 3 NUFFT" $\qquad$ $\nwarrow$ i) pre-multiply

Entire algorithm = three sequential steps $\qquad$ very simple, $<$ 10 lines of MATLAB

Cost $\mathcal{O}(N + M + \mathfrak{f}^2 \log \mathfrak{f})$ $\qquad$ In practice: 10^7 (sources+targets)/sec on laptop

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If targets on regular grid, use (faster) type 1 NUFFT   abbrev by "t3" and "t1"

Recommended software for NUFFTs on CPU: `FINUFFT`

`http://finufft.readthedocs.io`

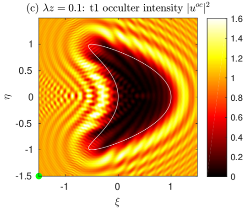
C++/OpenMP; beats others heartily (B et al '19)

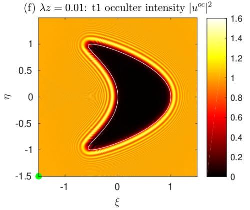
Results: target plane intensity pictures

Smooth kite test:

$\mathfrak{f} \approx 13$ (above), 130 (below):

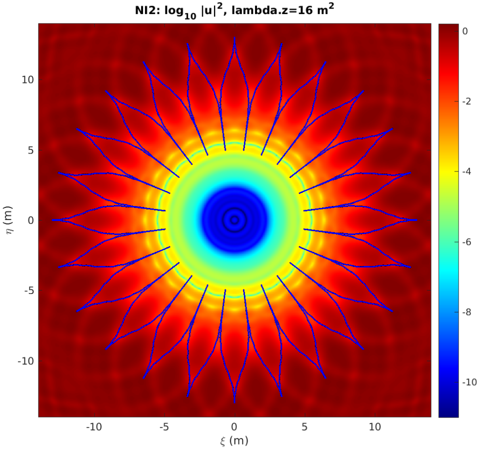
spectral acc. $< 10^{-12}$





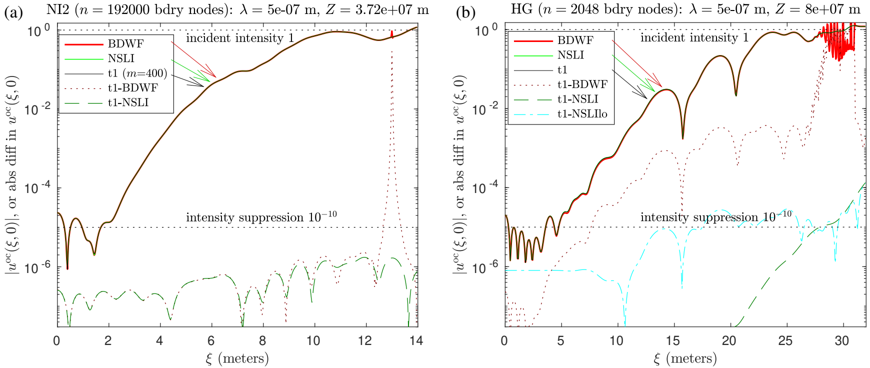
"NI2" design ideal starshade:

central deep shadow $|u^{oc}|^2 \sim 10^{-10}$, good!



Performance & validation for ideal starshades NI2, HG

Validation vs edge-integrals: (Cady '12; BDWF, from JPL SISTER codebase)



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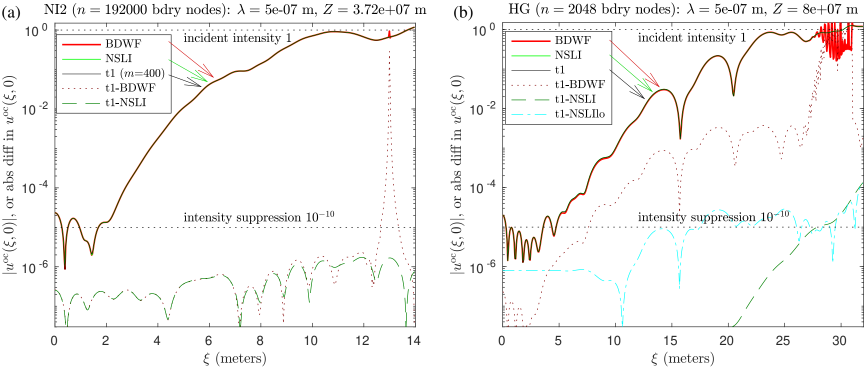


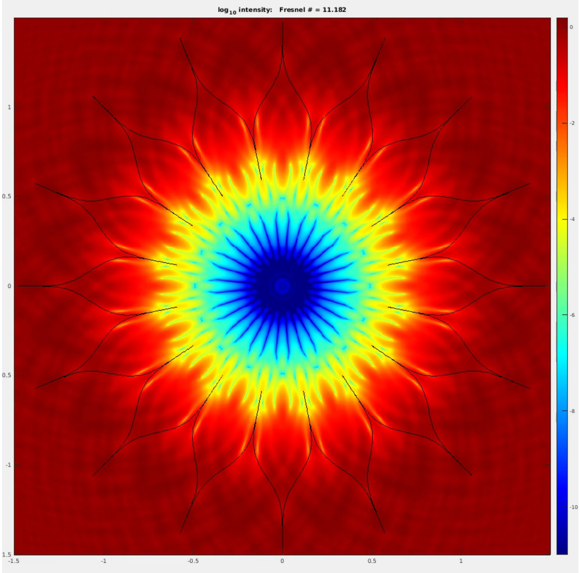
Speed against BDWF, for million-point target grid: i7 laptop

design	λ (m)	z (m)	$\mathfrak{f}$	m (petal)	total nodes	M (targets)	method	CPU time
NI2	5e-7	3.72e7	9.1	6000	n=192000	10^6, grid	BDWF	5361 s
				400	N=499200		NUFFT t1 (ε=10^{-8})	0.076 s
HG	5e-7	8e7	24	60	n=2048	10^6, grid	BDWF	80.5 s
				60	N=37440		NUFFT t1 (ε=10^{-8})	0.042 s

Conclusions: same accuracy reached, 3–5 orders of magnitude faster

Wavelength λ sweep movie

$M = 10^6$ targets, computed at $\sim$10 frames/sec (close to real time):



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Wavelength sweeps & modifications to JPL codes

Shadow depth study, 50 λ values, takes 6 seconds: including AQ gen; laptop

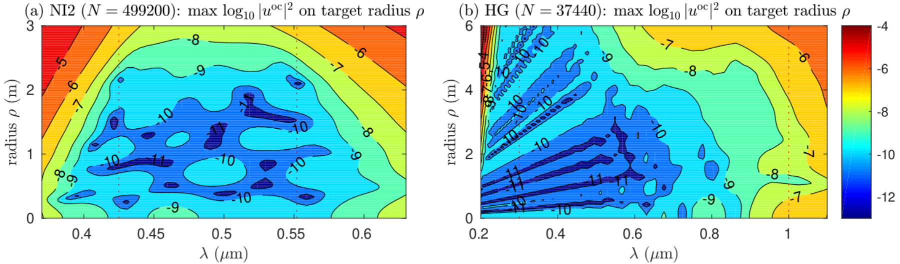

Fig 6 Intensity (on $\log_{10}$ scale indicated on the right) as a function of wavelength and target radius ρ from the center, for the two starshade designs (NI2 and HG) of Fig. 5. At each of 200 ρ values, the maximum over 300 angles is taken. The indicent intensity is 1. The NUFFT t3 method is used. Vertical dotted lines show the designed λ range.

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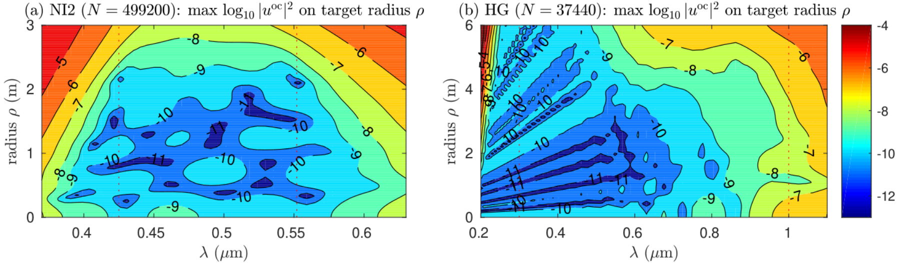

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Replacing BDWF by t3 in "PSF basis" task: hack JPL's code, proof-of-principle

- 3149 shifts of 16×16 pupil grids $\to$ key: do all targets at once!
- reduces JPL's MATLAB run-time from 6.5 hours to 2.6 seconds

Fun demo of complicated aperture: Koch snowflake

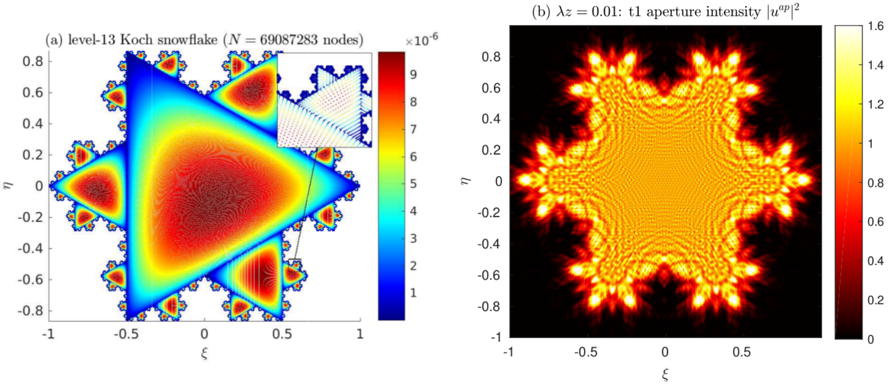

Fig 7 Koch fractal aperture diffraction example from Sec. 3.3. (a) shows the areal quadrature constructed by a union of about 67 million triangles. The color of each node (x_j, y_j) indicates its weight w_j using the scale on the right. The inset shows a zoom into the region shown, resolving individual nodes. (b) shows intensity (on $\log_{10}$ scale indicated on the right) computed on a million-point grid by the NUFFT t1 method in under 5 seconds.

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Conclusions

Fast method for accurate ($< 10^{-6}$) hard-edged Fresnel diffraction

$\sim 10^7$ pupil plane targets/sec, almost indep of starshade geom.

Excels when many targets: in practice $10^4\times$ *faster than previous methods*

Simple: 2D NUFFT + high-order areal quadrature for occulter domain

Future:

- shape variation/roughness started (JPL users: Dumont, Shaklan, Harness)
- continuous phase variation? trivial
- "0-1" aperture design probs: coronagraphs, zone plates. . .
- Question: can apply NUFFT to edge-integral, bypassing AQ?

"Efficient high-order accurate Fresnel diffraction via areal quadrature and the nonuniform FFT," A. H. Barnett, *J. Astron. Telesc. Instrum. Syst.* **7**(2), 021211 (21 pages), 2021. `arxiv:2010.05978`.

Code/docs: `https://github.com/ahbarnett/fresnaq`