Using the impedance-to-impedance map for frequency-domain scattering problems

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November 21, 2012

Abstract

If a body's interior Dirichlet-to-Neumann map is known at a given frequency, then the scattering problem from that body may be solved at that frequency. However, this map does not exist at certain interior resonant frequencies, causing a non-physical breakdown in the numerical scheme. This is fixed by instead using the impedance-to-impedance map. For the case of the body being a domain with constant interior wavenumber differing from the exterior free-space wavenumber (i.e. the transmission problem), we show the construction of the interior and exterior DtN and ItI maps in terms of boundary integral operators. Also in that case, for a smooth domain in $\mathbb{R}^2$, We test the resulting approaches to scattering numerically with dense linear algebra. We show that the ItI map fixes the resonance problem. Code is included. However, the standard approach using the exterior map gives a linear system that is not 2nd kind. We present alternative formulations in the DtN and ItI cases that are 2nd kind—this includes tests of Adrianna's new formulations from Oct 2012.

1 Interior Dirichlet-to-Neumann map

Let Ω be a bounded domain, and b be a real function with support in Ω. The interior Dirichlet BVP

$$[\Delta + \kappa^2(1 - b(x))]u = 0 \qquad \text{in } \Omega \tag{1}$$

$$u = h \qquad \text{on } \partial\Omega \tag{2}$$

has a unique solution for all h at all but a discrete set of real κ^2, which we call Dirichlet eigenvalues of Ω.

Let κ^2 not be a Dirichlet eigenvalue of Ω. Then the Dirichlet-to-Neumann operator Λ exists, and writing $\kappa^2(x) = (1 - b(x))\kappa^2$, if u is any solution to $(\Delta + \kappa^2(x))u = 0$ in Ω, then

$$\Lambda u|_{\partial\Omega} = u_n \tag{3}$$ {dtn}

where u_n is u's normal derivative on $\partial\Omega$.

Λ is self-adjoint when κ is real, since, for any functions u and v satisfing $(\Delta+\kappa^2(x))u = 0$ in Ω and $(\Delta + \kappa^2(x))v = 0$ in Ω, we have by Green's 2nd identity,

$$0 = \int_\Omega \overline{v}(\Delta + \kappa^2(x))u - u(\Delta + \overline{\kappa^2(x)})\overline{v} = \int_{\partial\Omega} \overline{v}u_n - u\overline{v_n} = (v|_{\partial\Omega}, \Lambda u|_{\partial\Omega}) - (\Lambda v|_{\partial\Omega}, u|_{\partial\Omega})$$

If $\partial\Omega$ is smooth, Λ is a ΨDO of order $+1$, and compact inverse, so has eigenvalues accumulating at ∞. In addition, it has a pole at each Dirichlet eigenvalue of Ω, and a zero at each Neumann eigenvalue of Ω.

1.1 Construction of constant-wavenumber interior and exterior maps by boundary integral operators

Here let b be constant in Ω, i.e. constant wavenumber κ. A physical example is that Ω is a lump of dielectric. Let $\mathcal{S}$, $\mathcal{D}$ be the usual single- and double-layer representations for the (constant wavenumber κ) Helmholtz equation on $\partial\Omega$. That is, if σ is a density function on $\partial\Omega$,

$$
\begin{aligned}
(\mathcal{S}\sigma)(x) &:= \int_{\partial\Omega} \Phi_\kappa(x,y)\sigma(y)ds_y && (4)\\
(\mathcal{D}\sigma)(x) &:= \int_{\partial\Omega} \frac{\partial\Phi_\kappa(x,y)}{\partial n_y}\sigma(y)ds_y && (5)
\end{aligned}
$$

where Φ_κ is the fundamental solution obeying $-(\Delta+\kappa^2)\Phi(x,\cdot)=\delta_x$. Let S, D be their restrictions to operators in $C(\partial\Omega)\to C(\partial\Omega)$, where in the case of D the integral is meant in the principal value sense.

The interior Green's representation formula says

$$u(x) = (\mathcal{S}u_n)(x) - (\mathcal{D}u|_{\partial\Omega})(x) \qquad x\in\Omega \tag{6}$$ {grf}

Taking $x\to\partial\Omega$ from inside and using the jump relations gives

$$Su_n = (D+\tfrac{1}{2})u|_{\partial\Omega} \tag{7}$$ {sudu}

Thus

$$\Lambda = S^{-1}(D+\tfrac{1}{2}) \tag{8}$$ {dtnsd}

Here the right factor captures the nullspace at Neumann eigenvalues, while S^{-1} blows up at each Dirichlet eigenvalue. Other representations exist (see DtN wiki of Demanet–Barnett), including

$$\Lambda = (D^*+\tfrac{1}{2})S^{-1} \tag{9}$$ {dtnds}

which follows via the Calderón identity $DS=SD^*$, or via an indirect formulation. Another is,

$$\Lambda = (D^*-\tfrac{1}{2})^{-1}T \tag{10}$$ {dtndt}

where T is the hypersingular operator with kernel $\partial^2\Phi_\kappa(x,y)/\partial n_x\partial n_y$. This is derived by taking the normal derivative of (6) on the inside boundary.

The exterior (radiative) Green's representation formula has an overall sign change from (6) [2]. The sign of the $\frac{1}{2}$ in the jump relation also changes. Thus the free space DtN for radiative fields in the exterior domain $\mathbb{R}^d\backslash\overline{\Omega}$ is

$$\Lambda_e = S^{-1}(D-\tfrac{1}{2}) = (D^*-\tfrac{1}{2})S^{-1}\ . \tag{11}$$ {dtnesd}

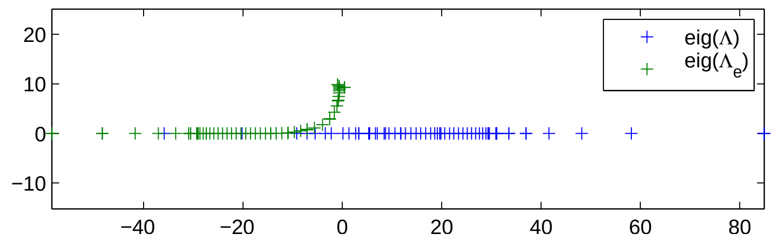

Figure 1: Numerically computed spectra of Λ and Λ_e with $N = 100$ boundary nodes. As N increases, eigenvalues are added extending to ∞ on both sides. f:dtn

Note that here the operators should be at the exterior wavenumber. Also, note that, although unbounded, Λ_e always exists for each κ_e, since the exterior Dirichlet BVP has a unique solution.

In `MPSpack` we may numerically approximate Λ and Λ_e on a smooth 2D star-shaped trefoil domain in the free space case (no dielectric, i.e. $b \equiv 0$), and check their spectra, as follows:

```
N = 100; s = segment.smoothstar(N, 0.3, 3);  % trefoil w/ periodic nodes
k = 10; o.qtype = 'm';                       % wavenumber, Kress quadr
S = layerpot.S(k,s,[],o); D = layerpot.D(k,s,[],o);
DtN = inv(S) * (eye(N)/2 + D);
DtNe = inv(S) * (-eye(N)/2 + D);
figure; plot([eig(DtN) eig(DtNe)],'+');
axis equal; legend('eig(\Lambda)','eig(\Lambda_e)');
```

This produces Fig. 1. Note that Λ is self-adjoint but that Λ_e is not (there are $O(\kappa)$ modes which generate waves with significant absorption at infinity).

2 Interior Impedance-to-Impedance map

Proposition 1. *Let $\eta \in \mathbb{C}$, $\operatorname{Re}\eta \neq 0$. Then the interior Robin BVP*

$$[\Delta + \kappa^2(1 - b(x))]u = 0 \qquad \text{in } \Omega \tag{12}$$

$$u_n + i\eta u|_{\partial\Omega} = f \qquad \text{on } \partial\Omega \tag{13}$$

has a unique solution for all real κ. {p:robin}

Proof. Existence follows from usual elliptic theory. For uniqueness consider u a solution for $f \equiv 0$. Then using Green's 1st identity, the boundary condition and the PDE, gives

$$-i\eta \int_{\partial\Omega} |u|^2 = \int_{\partial\Omega} \overline{u} u_n = \int_\Omega |\nabla u|^2 - \kappa^2(x)|u|^2.$$

Taking the imaginary part shows u, and hence u_n, vanishes on $\partial\Omega$, hence $u \equiv 0$ in Ω by unique continuation. ∎

Note that this idea that impedance boundary conditions lead to unique interior solutions, and hence a good coupling scheme between finite elements, is present in the work of Monk–Wang [3].

Now fix $\eta \neq 0$ (the right choice is probably $\eta = \kappa$), and let

$$f := u_n + i\eta u|_{\partial\Omega} \quad (14)$$
$$g := u_n - i\eta u|_{\partial\Omega} \quad (15)$$

be Robin traces of the field u. By analogy with the case of Ω a halfspace, we interpret f, g as wave amplitudes of u entering, leaving the domain respectively. Define the ItI operator R by

$$Rf = g \quad (16)$$

Proposition 2. *R exists for all κ, and is unitary for η real.*

Proof. Existence follows from Proposition 1. For unitarity, we insert (3) into $R(u_n + i\eta u|_{\partial\Omega}) = u_n - i\eta u|_{\partial\Omega}$ to get

$$R(\Lambda + i\eta)u|_{\partial\Omega} = (\Lambda - i\eta)u|_{\partial\Omega}$$

which holds for any data $u|_{\partial\Omega}$. Thus we have as operators,

$$R = (\Lambda - i\eta)(\Lambda + i\eta)^{-1} \ . \quad (17) \quad \{\texttt{itidtn}\}$$

One may check unitarity from this formula using e.g. $(\Lambda + i\eta)^* = (\Lambda - i\eta)$ □

Other properties of R: since it is unitary, its eigenvalues are on the unit circle. From (17) and the properties of Λ, we have that eigenvalues accumulate only at 1. We prove $R = \mathrm{Id} + \text{compact}$ by looking at (18) below and using that D and S are cpt.

2.1 Construction of constant-wavenumber maps by boundary integral operators

Inserting $2u_n = f + g$ and $2i\eta u|_{\partial\Omega} = f - g$ into (7) gives

$$(i\eta S - D - \tfrac{1}{2})f = (-i\eta S - D - \tfrac{1}{2})g$$

which holds for all solutions u, hence

$$R = (\tfrac{1}{2} + D + i\eta S)^{-1}(\tfrac{1}{2} + D - i\eta S) \quad (18) \quad \{\texttt{itisd}\}$$

This formula can also be derived by swapping factors in (17), inserting (8), and cancelling S. Likewise, (17) inserting (9) and cancelling S gives

$$R = (\tfrac{1}{2} + D^* - i\eta S)(\tfrac{1}{2} + D^* + i\eta S)^{-1} \quad (19) \quad \{\texttt{itids}\}$$

In an analogous manner to (10), we can also derive

$$R = [T - i\eta(\tfrac{1}{2} - D^*)]^{-1}[T + i\eta(\tfrac{1}{2} - D^*)] \ . \quad (20)$$

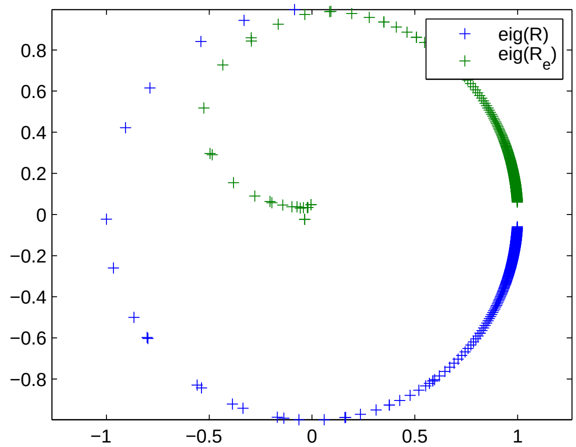

Figure 2: Numerically computed spectra of R and R_e. $N = 500$ boundary nodes were used to show accumulation of the spectra at $+1$. f:iti

An ItI operator also exists for radiative fields in the exterior of a domain. This exterior free-space ItI operator R_e may be written as above but with the signs of both terms $\frac{1}{2}$ changed, namely

$$R_e = (\tfrac{1}{2} - D - i\eta S)^{-1}(\tfrac{1}{2} - D + i\eta S) = (\tfrac{1}{2} - D^* + i\eta S)(\tfrac{1}{2} - D^* - i\eta S)^{-1} \qquad (21)$$ {itiesd}

Again, here the exterior wavenumber must be used for the operators.

Again we compute operators and their spectra numerically, continuing the above `MPSpack` code as follows:

```
eta = k;
ItI = inv(eye(N)/2+D+1i*eta*S) * (eye(N)/2+D-1i*eta*S);
ItIe = inv(eye(N)/2-D-1i*eta*S) * (eye(N)/2-D+1i*eta*S);
figure; plot([eig(ItI) eig(ItIe)],'+'); axis equal; legend('eig(R)','eig(R_e)');
```

This produces Fig. 2. Note that R is unitary but that R_e is not (there are $O(\kappa)$ modes which generate waves with significant absorption at infinity).

3 Direct methods for solving the scattering problem

3.1 Resonance problem with Dirichlet-to-Neumann maps

As Gunnar notes, if a region's interior DtN Λ is known, computing the exterior free space DtN Λ_e is one route to solving the overall scattering problem.

Let the exterior wavenumber be κ_e. Let the incident wave u^i solve $(\Delta + \kappa_e^2)u^i = 0$ in $\mathbb{R}^d \backslash \overline{\Omega}$ (this is a special case of Gunnar's v in the limit of infinitely distant source region). We have the representation $u + u^i$ in $\mathbb{R}^d \backslash \overline{\Omega}$, where u is the unknown (radiative) scattered field which thus obeys $\Lambda_e u|_{\partial\Omega} = u_n$ on $\partial\Omega$. By continuity of values and normal derivatives on $\partial\Omega$ we write the condition that the fields approaching $\partial\Omega$ from the inside satisfy the correct interior map,

$$\Lambda(u^i + u)|_{\partial\Omega} = u_n^i + u_n \ .$$

Substituting for u_n gives the equation for the unknown $u|_{\partial\Omega}$,

$$(\Lambda - \Lambda_e)u|_{\partial\Omega} = u_n^i - \Lambda u^i|_{\partial\Omega} \tag{22}$$ {dtnsc}

This is the same as Gunnar's (7.5). Once $u|_{\partial\Omega}$ is known, we can reconstruct $u(x)$ for $x \in \mathbb{R}^d \backslash \overline{\Omega}$ by computing $u_n = \Lambda_e u|_{\partial\Omega}$ and using the exterior version of the GRF (6).

Remark 3. *We don't know the nature of the operator $\Lambda - \Lambda_e$. It seems unbounded with pseudo order $+1$, by numerical evidence, and by $\kappa_e = \kappa$ case where it simplifies to S^{-1}, which is unbounded. This is because the operators are defined with the same normal sense, but one is for interior and other exterior, making the high-frequency (evanescent wave) parts which control the order of the operator* add *rather than cancel. (Conversely, $\Lambda + \Lambda_e$ where the high-freq parts do cancel, appears bounded.)*

Note that in the case $\kappa = \kappa_e$, which causes no scattering, the RHS vanishes because u^i also solves the interior problem, and the operator $\Lambda - \Lambda_e = S^{-1}$.

Let us demonstrate failure of this scheme at Dirichlet eigenvalues of the interior domain. We test the case of fixed interior wavenumber κ, so Λ is built from (8) at wavenumber κ, whilst Λ_e built from (11) at wavenumber $\kappa_e = 5$. This is done in code `fig_sweepk.m` which first searches for Dirichlet eigenvalues the trefoil domain, finding $\kappa_D = 10.5107955948496$ is the square-root of such an eigenvalue. In Fig. 3 we show error in scattered field u at the point $(1, 1)$ relative to the value computed by the standard Müller–Rokhlin scheme for the transmission problem [1] with Kress quadrature. The latter scheme is known to be robust at all wavenumbers. It is clear that the accuracy scales inversely with the distance from resonance, reaching $O(1)$ on resonance. One can check that this is explained by the condition number of $(\Lambda - \Lambda_e)$ diverging as the inverse of distance from resonance.

3.2 Fixing it with impedance-to-impedance maps

We repeat the experiment instead using ItI maps. By an identical derivation as (22), we get

$$(R - R_e)f = g^i - Rf^i \tag{23}$$ {itisc}

where f^i and g^i are the Robin data from the incident wave u^i, using the *same* η as for the internal problem (in experiments we fix $\eta = \kappa$). (If there were a huge material contrast at the interface $\partial\Omega$ then a new choice of η might be better.) Notice that the unknown is f, a Robin trace of the scattered field; once it is known one may reconstruct $g = R_e f$ then gets $u|_{\partial\Omega}$ and u_n from f and g. As above the exterior GRF is then used to get u at any point.

The result is shown by the lower green data in Fig. 3; as expected, errors are uniformly at machine precision around the resonance. For $N = 140$ the condition number of the linear system (23) is about 7 uniformly in k in this range. Note that we expect this to grow linearly in N; we observe it has reached only 22 by $N = 640$.

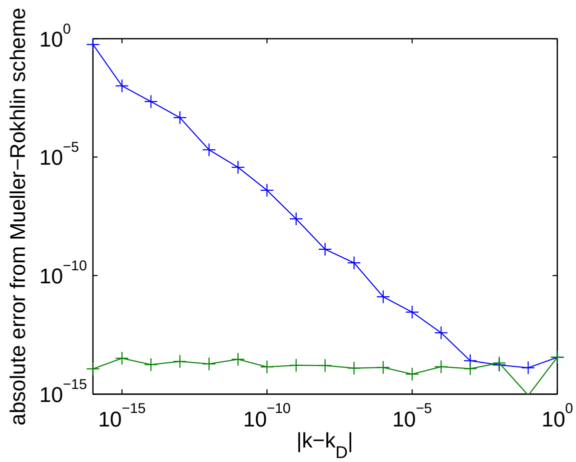

Figure 3: Numerical error in the scattered field at the point $(1,1)$ vs wavenumber "distance" from κ_D, the square-root of a Dirichlet eigenvalue. $N = 140$. The upper blue curve is for the Dirichlet-to-Neumann method, and the lower green the impedance-to-impedance. f:ksweep

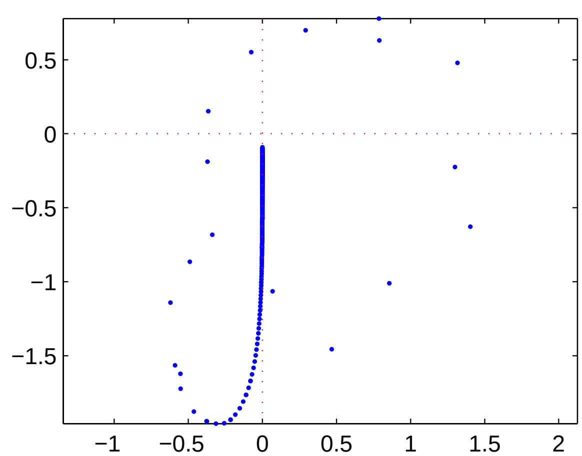

Figure 4: Spectrum of the numerical approximation to the operator $R - R_e$ appearing in (23), for $N = 640$. f:idiffspec

3.3 Conditioning of the 1st-kind linear system

In Fig. 4 we plot the eigenvalues for a large value $N = 640$ to gain partial insight into the conditioning. It appears that eigenvalues accumulate slowly at 0, approaching from below. Theoretically we have that R and R_e are Id + compact, thus their difference is compact. Thus the equation (23) is not 2nd-kind, and for large N (demanded by large κ) this would start to lose digits, and is known to be a Bad Idea (TM) if you are a 2nd-kind purist. However, for low frequencies tested, condition number remains very small, around 20 for N several hundred.

3.4 Construction of free-space exterior DtN or ItI map

In Gunnar's preprint the construction of exterior DtN Λ_e (he calls S) is suggested by BIE methods. The un-accelerated nested dissection construction of an interesting interior body's R or Λ is $O(N^3)$ where N is number of boundary nodes. Constructing the exterior map via dense linear algebra and boundary integral operators as above would also take $O(N^3)$, and would be acceptable. However, the HBS-accelerated dissection solver is $O(N^2)$, since N^2 is the number of interior unknowns; we thus seek an $O(N^2)$ method for exterior maps.

4 2nd kind formulations for solving the scattering problem

Here's a summary of methods come up by Alex and Adrianna. Alex spent in June 2012 a while on paper trying all direct and indirect representations of u, the scattered field in $\mathbb{R}^d \backslash \overline{\Omega}$. By that we mean: direct represenions have $u|_{\partial\Omega}$ or u_n as the unknown (as in (22)); indirect representations have u generated as the potential due to an unknown density function. Adrianna came up with some new formulations in Oct 2012.

4.1 Dirichlet-to-Neumann map case

We return to the (non-robust) case of using Λ.

4.1.1 Alex's DtN formulation

The two possible direct formulations give equations with operator $(\Lambda - \Lambda_e)$ in the case of $u|_{\partial\Omega}$ unknown, or $(I - \Lambda\Lambda_e^{-1})$ in the case of u_n unknown. I think the latter is not 2nd kind.

However, the indirect SLP formulation $u = \mathcal{S}\sigma$ in $\mathbb{R}^d \backslash \overline{\Omega}$ gives a 2nd kind equation as follows,

$$(D^* - \tfrac{1}{2} - \Lambda S)\sigma = -u_n^i + \Lambda u_i \qquad (24) \quad \{\texttt{dtnsc2}\}$$

Notice that S is order -1, while Λ is order $+1$. Therefore their product is bounded when Λ exists. Moreover, we have that $\Lambda S = \frac{1}{2}$ + compact, since in the case of interior wavenumber equal to that of the exterior, using (10) and the Calderón identity $TS = D^{*2} - \frac{1}{4}$, we get

$$\Lambda S = (D^* - \tfrac{1}{2})^{-1} TS = (D^* - \tfrac{1}{2})^{-1}(D^{*2} - \tfrac{1}{4}) = D^* + \tfrac{1}{2}$$

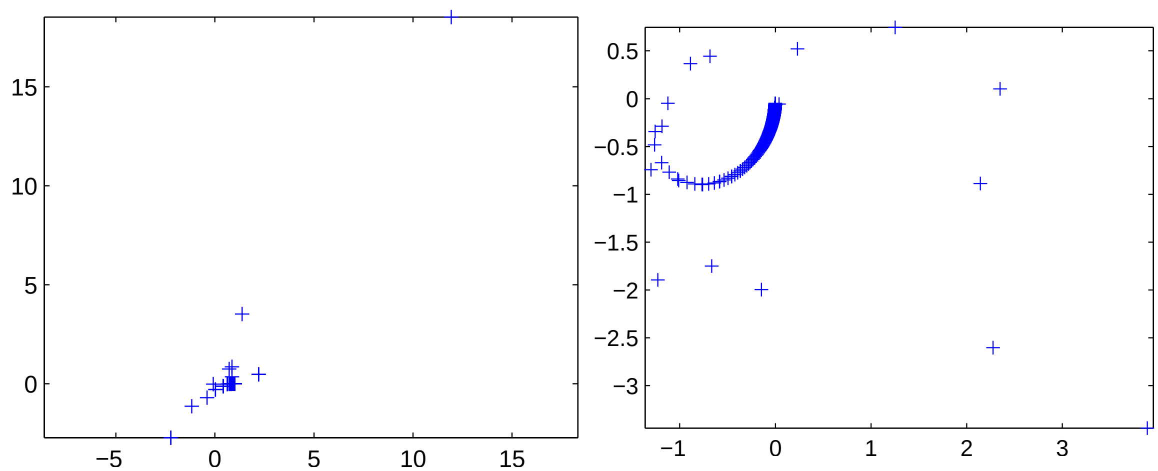

Figure 5: Left: Spectrum of the numerical approximation to the operator $A = \frac{1}{2} - D + S\Lambda$ appearing in (25), for $N = 300$. Accumulation at 1 is apparent. Right: Spectrum of the numerical approximation to the operator $A = -SRB + D - \frac{1}{2} - i\eta S$ appearing in (28), for $N = 600$. Accumulation at 0 is apparent. f:adrianna

which is explicitly of the claimed form when $\partial\Omega$ has no corners. For more general interior wavenumbers Λ only varies from the free-space case by a compact perturbation (e.g. since $T_\kappa - T_{\kappa'}$ is only weakly singular), thus $\Lambda S = \frac{1}{2}$ + compact. I checked this numerically and it's true. I believe this will apply to general variable interior wavenumber too. Thus (24) is 2nd-kind because the two terms of $-\frac{1}{2}$ add to give $-I$.

The indirect DLP formulation $u = \mathcal{D}\tau$ gives a hypersingular equation which can be regularized by Calderón to give something equivalent to the SLP case. Mixtures of SLP + DLP don't seem to give anything new.

4.1.2 Adrianna's DtN formulation

Adrianna had the idea of left- or right-regularizing the direct formulations. Take (22) with Λ_e given by the first form in (11), then left-multiply by S, gives

$$(\tfrac{1}{2} - D + S\Lambda)u|_{\partial\Omega} = S(u_n^i - \Lambda u^i)\ . \qquad (25)$$ {dtnsc3}

The operator $A = \frac{1}{2} - D + S\Lambda$ is the transpose of that in the indirect formulation (24), up to sign. The spectral properties are thus the same. $A = \text{Id} + \text{cpt}$, since in the free-space case (same κ inside as out), using (8), $S\Lambda = \frac{1}{2} + D$ and D is cpt.

We test this gives 13-digit agreement with Müller–Rokhlin for $\kappa = 10$, $\kappa_e = 5$, in `demoItI.m` with formulation `form='a'`. As N grows, A converges to condition number 397, which is a bit high, but this is due to a single small singular value. Spectrum of A in Fig. 5(a)

4.2 Impedance-to-impedance map case

This is what we care about, since it is robust.

4.2.1 Alex's ItI formulation

The two possible direct formulations are covered by the cases of taking f as the unknown as in (23), in which case the operator in the linear equation is $(R - R_e)$, or g as the unknown, in which case it's $(I - RR_e^{-1})$. Both operators are compact, thus the equations are not 2nd kind. The underlying reason is that f and g have the same amount of u_n present, and this is dominant at high spatial frequencies on $\partial\Omega$. Thus R tends to I at high frequencies, hence is of the form $I+$ compact.

Neither obvious indirect formulation gives 2nd kind. Take the SLP $u = \mathcal{S}\sigma$ in the exterior. Then the boundary condition $g - Rf = -g^i + Rf^i$ gives

$$[D^* - \tfrac{1}{2} - i\eta S - R(D^* - \tfrac{1}{2} + i\eta S)]\sigma = -g^i + Rf^i$$

which has a compact operator in the square brackets. This exemplifies the problem: since f and g have the same amount of u_n, any identity operators created by jump relations cancel. The DLP gives hypersingular or can be Calderón-regularized to the above with σ replaced by $(D^* + \frac{1}{2})\tau$, no different.

The cancellation failures of the above suggest that an operator different from R has to be used—one which instead changes the order of the derivative, i.e. which has pseudodifferential order different from 0. Consider the new *compact* operator P,

$$P = (R - I)/2, \qquad R = I + 2P, \qquad P(u_n + i\eta u|_{\partial\Omega}) = -i\eta u|_{\partial\Omega} \tag{26}$$ {P}

Note that in going from Robin to pure Dirichlet data, the order is -1, just like Λ^{-1}. Also, P is bounded and exists for all interior wavenumbers, since R is. The indirect DLP $u = \mathcal{D}\tau$ with the boundary condition $-i\eta u|_{\partial\Omega} - Pf = +i\eta u^i|_{\partial\Omega} + Pf^i$ gives

$$[-i\eta D - i\eta\tfrac{1}{2} - P(T + i\eta D + i\eta\tfrac{1}{2})]\tau = +i\eta u^i|_{\partial\Omega} + Pf^i \tag{27}$$ {itisc2}

We also have using Calderón and $P = (\frac{1}{2} + D + i\eta S)^{-1}S$ that $PT = -\frac{1}{2}$ + compact. Therefore (27) is 2nd kind with the multiple of the identity being $\frac{1}{2}(1 - i\eta)$, so 2nd kind unless $\eta = -i$.

Numerical implementation of the above requires constructing P (easy since you have R), but also applying the T operator on $\partial\Omega$. (To test this I need to first code the cot term into the T matrix evaluation in `MPSpack`.) With corners, it could be bad.

The indirect SLP gives a 1st-kind equation since all is compact.

As with DtN, mixtures of SLP + DLP don't seem to give anything new.

4.2.2 Adrianna's ItI formulation—not 2nd kind

We write a new form for the exterior ItI map, substituting the first form of (11) into the exterior (identical) version of (17),

$$R_e = (S^{-1}(D - \tfrac{1}{2}) - i\eta)(S^{-1}(D - \tfrac{1}{2}) + i\eta)^{-1} = (S^{-1}(D - \tfrac{1}{2}) - i\eta)B^{-1}$$

where $B := \Lambda_e + i\eta = S^{-1}(D - \frac{1}{2}) + i\eta$ is bounded for each κ_e since Λ_e always exists. As usual, S and D are for the exterior wavenumber. Substitute this in the direct formulation (23) and left-multiply by $-S$ to get

$$[-SR + (D - \tfrac{1}{2} - i\eta S)B^{-1}]f = S(-g^i + Rf^i)$$

Finally let $\hat{f} = B^{-1}f$, then

$$\left(-SRB + D - \tfrac{1}{2} - i\eta S\right)\hat{f} = S(-g^i + Rf^i) \qquad (28) \quad \{\texttt{itisc3}\}$$

The data f is reconstructed via $f = B\hat{f}$.

What's happening spectrally? B is unbounded, but $-SRB = \frac{1}{2}+$ cpt, we believe. This is suggested by the following calculation in the case of free-space, $\kappa_e = \kappa$, by substituting the interior R from (17) and Λ from (8) to get, after a bunch of shifting S around,

$$-SRB = (\tfrac{1}{2} + D - i\eta S)R\ .$$

Since $R = \mathrm{Id} +$ cpt, and D and S are cpt, this supports the claim. We'd need a perturbation argument for general κ inside (not sure how to prove). This means that the multiples of Id in the operator $A = -SRB + D - \frac{1}{2} - i\eta S$ unfortunately cancel, so A is cpt. This is not 2nd kind.

Testing (28) gives 13-digit agreement with Müller–Rokhlin, and cond(A) which grows weakly, reaching 200 by $N = 1000$. Spectrum of A in Fig. 5(b) confirms it's not 2nd-kind. See code `demoItI.m` with `form='ia'`.

Can we find a 2nd-kind variant?

5 Discussion, open issues (out of date)

We found a ItI scheme that tests well and even though it's 1st kind the condition number grows remarkably slowly (we expect conditioning to grow like N).

We did find a 2nd kind ItI map scheme, which involves applying the T operator, and didn't yet test this numerically.

1. Corners. The nested construction is based on squares (I am somewhat amazed that this works given the unknown corner singularities in Λ). My above numerical tests are for smooth boundary curves. I don't know how it will be meshing two totally different methods for dealing with a square boundary and coupling them together. I expect trouble.

2. Operator equation (23) is not 2nd-kind. Not a good idea for conditioning, although in practice for N up to several hundred, and $\kappa < 10^2$, the condition number is no more than 20. So maybe good enough for first scattering paper using it. Or, it may be worth presenting (27) also, if we can implement it, to see if helps in practice (I expect that applying T may lose digits just as badly as having a weakly-growing condition number).

3. I actually believe that a compressed representation of the *exterior* DtN map would find a huge application area in FEM scattering where a radiation condition is applied on an artifical enclosing surface. People have spent a lot of time on various local approximations, absorbing BCs, PMLs, etc.

4. What happens when there is a physical near-resonance? E.g. $b(x)$ has a negative smooth depression supporting whispering-gallery modes which are exponentially long-lived? As Gunnar says, we can't expect the method to be better conditioned than the physical problem!

References

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