Puzzler answer

based on reply from Jonathan Goodman, Courant Institute, NYU

March 22, 2005

Defining

$$G := \int_{-\infty}^{\infty} f(t)w(t)dt - \bar{f}, \tag{1}$$

then its expected value is $E[G] = 0$. The variance can be written using the definition of C_f as

$$E[G^2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w(t_1)w(t_2)C_f(t_2 - t_1)dt_1 dt_2. \tag{2}$$

Defining Fourier transforms

$$\tilde{w}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}w(t)dt \tag{3}$$

and

$$\tilde{C}_f(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\tau}C_f(\tau)dt \tag{4}$$

gives

$$E[G^2] = \int_{-\infty}^{\infty} |\tilde{w}(\omega)|^2 \tilde{C}_f(\omega)d\omega. \tag{5}$$

The function $\tilde{w}(\omega)$ is peaked around $\omega = 0$ and much narrower than the spectral density $\tilde{C}_f(\omega)$. Taylor expanding gives

$$\tilde{C}_f(\omega) = \tilde{C}_f(0) + \frac{1}{2}\tilde{C}''_f(0)\omega^2 + \frac{1}{4!}\tilde{C}''''_f(0)\omega^4 + \cdots. \tag{6}$$

A simple bound is

$$|\tilde{C}''''_f(\omega)| \leq \frac{1}{2\pi}\int_{-\infty}^{\infty} \tau^4 |C_f(\tau)|d\tau, \tag{7}$$

which implies that the Taylor remainder term is bounded by const.ω^4 under normal circumstances (e.g. exponential decay of the covariance function). Then, we get

$$\begin{aligned} E[G^2] &= \tilde{C}_f(0)\int_{-\infty}^{\infty}|\tilde{w}(\omega)|^2 d\omega + \frac{1}{2}\tilde{C}''_f(0)\int_{-\infty}^{\infty}|\omega\tilde{w}(\omega)|^2 d\omega \\ &\quad + \frac{1}{4!}\tilde{C}''''_f(0)\int_{-\infty}^{\infty}|\omega^2\tilde{w}(\omega)|^2 d\omega + \cdots \end{aligned} \tag{8}$$

The first term is from the Kubo formula $D_f \int_{-\infty}^{\infty} |w(t)|^2 dt$, where diffusion rate is $D_f := \int_{-\infty}^{\infty} C_f(\tau) d\tau$.

The second term is the correction term you mentioned. The third term should be smaller under the hypotheses you suggest. For example, given the bound on $|C_f''''|$, it is less than a constant times (not assuming $w(t) > 0$ since fancy windows might not have this property) $\left[\int |w(t)| dt\right] \times \max |w''''(t)|$.

A way to make the discussion and your hypotheses on w more concrete is to suppose that there is an ur-window, $u(t)$, and that $w(t) = (1/L)u(t/L)$ and then to seek expansions/theorems in the limit $L \to \infty$ with the f process and $u(t)$ fixed. Then the main term (the Kubo term) has order $1/L$, as it should for using $O(L)$ "data points". The correction has order $1/L^3$.

In this limit there is a central limit theorem that says that G is approximately normal. We characterize a normal by its mean and variance, which are calculated above. There also are corrections to the normal approximations, which sometimes are called the Edgeworth expansion. The corrections depend on moments of G of higher than second order. The first correction depends on $E[G^3]$, which should be of order $1/L^2$ (if it's not zero) and is called skewness. The next one depends on $E[G^4 - 3G^2]$, which is kurtosis and should be of order $1/L^3$ and should not be zero in general.

I hope this helps.