Holiday puzzler on stochastic processes

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Here is a question which arose in my thesis work, on numerical estimation of the noise power spectrum (spectral density) of a time-dependent signal, using a smooth windowing function. The signal happened to derive from a numerical simulation of an ergodic dynamical system. Recently a random student in the UK asked me for a rigorous explanation (of equation (B.4) in my thesis [1] and I discovered I didn't know one. The issue is as follows.

Let the signal $f(t)$ be a stationary stochastic process with average value $\bar{f}$. Its autocorrelation

$$C_f(\tau) := \lim_{T\to\infty} \frac{1}{T} \int_0^T (f(t) - \bar{f})(f(t+\tau) - \bar{f}) dt \tag{1}$$

we assume dies exponentially on a characteristic time-scale t_0. By definition we have

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(t) dt = \bar{f}. \tag{2}$$

We want to generalize to integrating against a windowing function $w(t)$ with unit norm $\int_{-\infty}^{\infty} w(t) dt = 1$. When $w(t)$ is smooth and changes only on a timescale much longer than t_0, we expect the following,

$$\int_{-\infty}^{\infty} f(t) w(t) dt = \bar{f} + \text{correction terms}. \tag{3}$$

The question is, given $w(t)$, $\bar{f}$ and $C_f(\tau)$, what are the correction terms, or how do they scale? More precisely we'd want their expectation value over the ensemble from which f is drawn. Because of the separation of timescales, there is a multiscale analogy here to integrating away fast fluctuations.

Intuitively I expect if $C_f(\tau) = C_f(-\tau)$ then since odd moments vanish the first correction might scale like the typical size of $w''(t)$ times the second moment $\int_{-\infty}^{\infty} \tau^2 C_f(\tau) d\tau$, due to the curvature of w interacting with the correlation of $f(t)$. However, I don't know if such results exist, or where to find them.

Do you?

[1] See `http://www.cims.nyu.edu/~barnett/thesis/thesis.ps.gz`