Accurate Evaluation of Layer Potentials up to the Boundary

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Introduction

DIRICHLET BOUNDARY VALUE PROBLEM

The goal of our research is to estimate the numerical solutions to the Laplace's equation, i.e. given a domain Ω and its boundary $\partial\Omega$, find a continuous function u satisfying the following equations

$$\begin{cases} \Delta u = u_{xx} + u_{yy} = 0 \text{ in } \Omega \\ u = f \text{ on } \partial\Omega \text{ with a given function } f \end{cases}$$

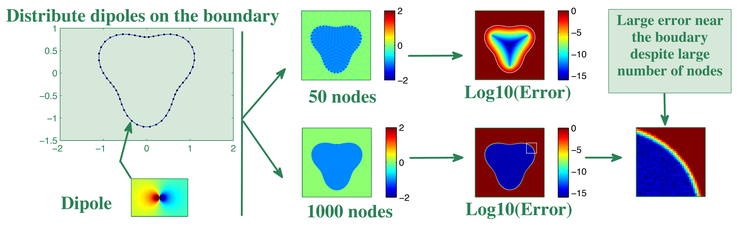
Applying potential theory [1], we represent the solution uniquely as a double-layer potential where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$:

$$u(\mathbf{x}) = \int_{\partial\Omega} \frac{\partial \Phi(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}_y} \tau(\mathbf{y}) ds_y \quad \text{where} \quad \Phi(\mathbf{x},\mathbf{y}) = \frac{1}{2\pi} \log \frac{1}{|\mathbf{x}-\mathbf{y}|}$$

For a given density function $\tau(\mathbf{y})$ on the boundary, we need to find an algorithm to evaluate u in the interior as accurately as possible.

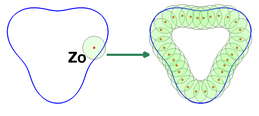
ERROR NEAR THE BOUNDARY

A prevalent method to evaluate u uses *discrete quadrature* to turn the integral into the sum of the *dipoles* on $\partial\Omega$, which is accurate on the inner region but yields unacceptable error near $\partial\Omega$.



TAYLOR EXPANSION

To resolve the aforementioned problem, we identify $\mathbb{R}^2$ with $\mathbb{C}$ and use the ***Taylor expansion*** centered at a point z_0 near $\partial\Omega$ to evaluate u locally instead of using quadrature on the entire domain.



$$u(\mathbf{x}) = Re\left(\sum_{n=0}^{T} c_n (z - z_0)^n\right) \quad \text{where} \quad \begin{cases} z = x_1 + ix_2 \\ \mathbf{x} = (x_1, x_2) \end{cases}$$

To evaluate the coefficient c_n, we introduce a boundary integral and Lagrange Interpolation. Now, from a small number of given values of u on $\partial\Omega$, say N, we can evaluate u accurately up to the boundary.

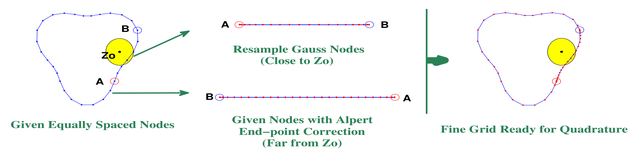
Evaluating the Taylor Coefficients

ALPERT CORRECTION & GAUSSIAN QUADRATURE

We can express the n^{th} Taylor coefficient as the following integral where $y(t)$, $0 \le t < 2\pi$ parametrizes $\partial\Omega$ in the complex plane

$$c_n = \frac{-1}{2\pi} \int_0^{2\pi} \frac{e^{i\gamma(y(t))}}{(y(t) - z_0)^{n+1}} \tau(y(t)) y'(t) dt$$

To evaluate c_n, we build a set of finely spaced nodes on $\partial\Omega$ consisting of Gauss nodes on AB and original nodes on BA with the endpoints handled by Alpert correction.[2]



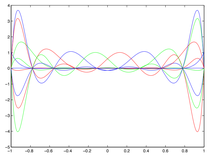
With the new grid, we apply Gaussian quadrature to estimate

$$\int_0^{2\pi} F(t)dt \approxeq \sum_{i=1}^{n} w_i F(t_i) \text{ where } w_i \text{ are known weights.}$$

Gaussian quadrature is guaranteed to converge uniformly for any continuous function.

LAGRANGE INTERPOLATION

With the new grid, we need to evaluate $\tau(t)$ at *any* point t given only the set of N points. Taking n local points t_i of density τ_i, $i \in \{1, 2, .., n\}$, we want to find a polynomial $p(t)$ that approximates $\tau(t)$, i.e. for all i, $p(t_i) = \tau_i$.



Lagrange polynomials on $[-1, 1]$ for $n = 10$

First, we define ***Lagrange polynomials***:

$$L_j(t) = \prod_{i=1, i\neq j}^{n} \frac{t - t_i}{t_j - t_i} \quad \text{so} \quad L_j(t_i) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

The interpolating polynomial can be written as

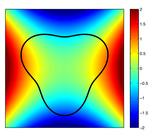
$$p(t) = \tau_1 L_1(t) + \tau_2 L_2(t) + ... + \tau_n L_n(t)$$

The equally spaced interpolation can sometimes diverge on the edges of the interval (e.g. *Runge's Phenomenon*). In our algorithm, we choose t_i such that the value of interest is at the middle of the interval where the interpolation is guaranteed to converge.

Result

We test our algorithm with a known function $u = x^2 - y^2$ and use *MPSpack* [3] to generate a density function τ for $N = 100$ points on the boundary.
Parameters for our evaluation method include:

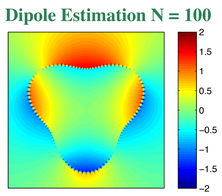
- $|z_0|$: radius of the Taylor expansion center
- δ: radius of the Taylor discs;
- N_{z_0}: number of points z_0.

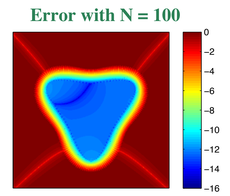


$u = x^2 - y^2$

Conventional Method

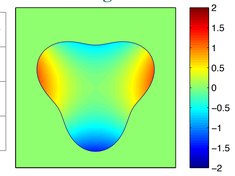
$\lvert z\rvert$	Error
0.6	8.55E-13
0.7	6.44E-11
0.8	2.43E-07
0.9	7.80E-04
0.95	4.11E-02

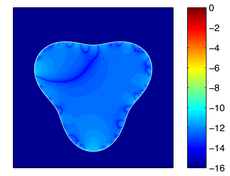




Our Method

δ	$\lvert z_0\rvert$	N_{z_0}	Max(Error)
0.15	0.7	30	2.48E-07
0.2	0.7	30	5.98E-09
0.2	0.7	40	6.52E-10





Notes:

- $\log_{10}$ (Error) is shown.
- All trials use Taylor degree $T = 30$, Gaussian quadrature with 200 points, interpolation order $P = 20$, and Alpert order $A = 20$.

Conclusion

Combining the mentioned techniques, we derive an algorithm to evaluate the numerical solutions of the Dirichlet Boundary Value Problem with high accuracy up to the boundary from a small number of given nodes.To ensure its accuracy, we need to test our method further on different u with singularities outside but near Ω.

References

[1] Kress, Rainer. Linear Integral Equations. Berlin: Springer-Verlag, 1989. Print.

[2] Alpert, Bradley. "High-order Quadratures for Integral Operators with Singular Kernels." Journal of Computational and Applied Mathematics 60.3 (1995): 367-78. Print.

[3] Barnett, Alex, and Timo Betcke. MPSpack. MATLAB Toolbox to solve Helmholtz PDE, wave scattering, and eigenvalue problems.