Mathematical Wave Dynamics: Theory summary 1

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1 Simple Harmonic Oscillator (SHO)

Paradigm system is mass m attached to spring strength k. Motion in 1d is scalar function $x(t)$. We have 1 *degree of freedom* (dof).

- Newton's 2nd Law: $m\ddot{x} = F$
- Hooke's Law (linear response): $F = -kx$, where $x = 0$ is the *equilibrium* position.

Gives linear, 2nd-order ordinary differential equation (ODE), $\ddot{x} + \frac{k}{m}x = 0$.
Substituting $x(t) = ae^{i\omega t}$ we find that only two roots $\omega = \pm\omega_0$ are possible if $a \neq 0$, where $\omega_0 = +\sqrt{k/m}$. Any linear combination of these two rotating complex exponentials is a solution. There are 3 common equivalent ways to write the *general solution*,

$$x(t) = \mathrm{Re}\left[ae^{i\omega_0 t}\right] \; = A\cos\omega_0 t + B\sin\omega_0 t \; = C\sin(\omega_0 t + \phi), \tag{1}$$

in a way that restricts to real (physical) solutions. $a \in \mathbb{C}$ and $A, B, C, \phi \in \mathbb{R}$.
This is called *harmonic motion.* Motion is periodic with period T, related to the *angular frequency* ω_0 by $\omega_0 = 2\pi f = 2\pi/T$.
C is *amplitude* (also a can be called a [complex] amplitude), and ϕ is *phase.*
Note smaller mass or more rigid spring both lead to higher frequency.

An *initial value problem* consists of finding the coefficients in a general solution which match a given position and velocity at *e.g.* time $t = 0$.

2 Normal Modes

A system described by N coordinates $\mathbf{x} \equiv \{x_i\} \equiv (x_1, x_2, \cdots x_N)$ has N dofs.
A *normal mode* is a solution where all coordinates move with the same period:

$$x_i(t) = a_i e^{i\omega t}. \tag{2}$$

Note: $i = \sqrt{-1}$ in exponential $\neq$ subscript i labeling coordinate.
Motion is *separable* into a product of a function a_i depending only on coordinate i and a function $e^{i\omega t}$ depending only on time t.

- Assume same mass for each coord, so $m\ddot{x}_i = F_i$
- Most general linear response is $F_i = -\sum_{j=1}^{N} K_{ij} x_j$, where K is an $N \times N$ matrix, generally symmetric ($K = K^T$).

Gives (coupled) ODE, in vector notation: $\ddot{\mathbf{x}} + \frac{1}{m} K\mathbf{x} = \mathbf{0}$.
Substituting Eq.(2) gives, $(K - m\omega^2 I)\mathbf{a} = \mathbf{0}$.

Nontrivial ($\mathbf{a} \neq \mathbf{0}$) solution only for discrete mode frequencies $\omega_n = +\sqrt{\frac{\lambda_n}{m}}$ where λ_n, $n = 1 \cdots N$ are the *eigenvalues* (also called *spectrum*) of K.
Each mode n must then have amplitude $\mathbf{a}_n$ proportional to an *eigenvector* $\mathbf{v}_n$ of K corresponding to eigenvalue λ_n.

- Linear algebra reminder: eigenvalue equation $K\mathbf{v}_n = \lambda_n \mathbf{v}_n$.
- K symm $\Leftrightarrow$ real λ_n, real orthogonal $\mathbf{v}_n$.
- choose unit-length eigenvectors so orthonormal, $\mathbf{v}_m \cdot \mathbf{v}_n = \delta_{mn}$
- *Kronecker delta* defined by $\delta_{mn} = 1$ if $m = n$, 0 if $m \neq n$.
- Note we ignored issues arising if λ's not distinct.

General solution is a linear combination of periodic normal modes but is itself generally *not periodic* (because the mode frequencies need not have rational ratios). General solution for each mode is as in Eq.(1), giving, in vector form,

$$\mathbf{x}(t) = \mathrm{Re}\left[\sum_{n=1}^{N} a_n e^{i\omega_n t} \mathbf{v}_n\right] = \sum_{n=1}^{N} C_n \sin(\omega_n t + \phi_n)\mathbf{v}_n. \tag{3}$$

Initial value problem: using the first form in Eq.(3), we want set $\{a_n\}$ given initial position $\mathbf{x}(0)$, velocity $\dot{\mathbf{x}}(0)$. We find,

$$a_m = \text{real part} + \text{imag part} = \mathbf{v}_m \cdot \mathbf{x}(0) - i\frac{1}{\omega_m}\mathbf{v}_m \cdot \dot{\mathbf{x}}(0) \tag{4}$$

Deriving this involved taking dot product of $\mathbf{v}_m$ with Eq.(3), and using the useful rule $\sum_m \delta_{mn} f_m = f_n$ where f is any function on the integers.

2.1 2-mass linear chain

In class we discussed the case of two masses m connected together and to fixed walls by three springs k, moving along a line. There are $N = 2$ dofs: x_1, x_2 are the horizontal displacement of each mass from equilibrium.
We showed that the spring matrix was $K = k\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$.
Dy *diagonalizing* K, we find the two modes are:

- $\lambda_1 = k$ giving $\omega_1 = +\sqrt{k/m}$ and $\mathbf{v}_1 = (1,1)$, so the masses move similarly.
- $\lambda_2 = 3k$ giving $\omega_2 = +\sqrt{3k/m}$ and $\mathbf{v}_2 = (1,-1)$, so the masses move opposite to each other.

Any general motion is a combination of these 2 modes, and we emphasize that it is generally not periodic.
See the course webpage for link to an applet which allows you to play with this system.