Senior Honors Class

Spring 2004 — Oliver Bühler & Alexander Barnett

Summary notes for lecture 2, 27 Jan 2004

2.1 Variational method

For given symmetric matrix $\mathsf{K} = \mathsf{K}^T$ and vector $\mathbf{v}$ define the *quadratic form*

$$Q = \mathbf{v}^T \cdot \mathsf{K} \cdot \mathbf{v}. \tag{1}$$

For example, in two dimensions $\mathbf{v} = (v_1, v_2)$,

$$\mathsf{K} = \begin{pmatrix} a & c \\ c & b \end{pmatrix}, \quad \text{and} \quad Q = av_1^2 + bv_2^2 + 2cv_1v_2. \tag{2}$$

For unit[1] vectors $\mathbf{v}$ (i.e. $|\mathbf{v}| = 1$) the quadratic form has the important property that it takes *extremal* values λ^* for unit vectors $\mathbf{v}^*$ that are the *eigenvectors* of K. Furthermore, the extremal values λ^* then are the corresponding eigenvalues, i.e.

$$\mathsf{K} \cdot \mathbf{v}^* = \lambda^* \mathbf{v}^*, \qquad Q(\mathbf{v}^*) = \lambda^*. \tag{3}$$

The proof is a straightforward application of Lagrange multipliers for constrained optimization.

The variational method is useful for eigenvalue prediction (and hence vibration frequency prediction) because it implies that estimating λ^* from an approximate unit eigenvector $\mathbf{v} = \mathbf{v}^* + \boldsymbol{\epsilon}$ gives an approximate eigenvalue $\lambda = Q(\mathbf{v}) = \lambda^* + O(|\boldsymbol{\epsilon}|^2)$. In other words, a linear error in $\mathbf{v}$ becomes a quadratic error in λ because of the extremal property of Q at the true eigenvector $\mathbf{v}^*$. This is an enormous improvement in accuracy for practical computations. The analytical procedure for computing vibration frequencies based on the variational method is called the *Rayleigh–Ritz* method. The analogous numerical procedure is called the *Finite-Element method*, which is widely used in engineering today.

Project suggestion: Use the Finite-Element method to study the vibrations of an interesting object (e.g. bike, train, bridge, space station…).

2.2 Continuum limit for linear chain

Consider N masses with common weight m that are indexed by $i = 1 \ldots N$ and connected by springs with common k; don't worry about the end points. Let the numbers $u_i(t)$ measure the displacement of each mass from its rest position at time t, i.e. $u_i = 0$ at rest. Therefore, the quantity $u_{i+1} - u_i$ is the extension of the spring between the masses i and $i + 1$. Newton's law is

$$m\ddot{u}_i = k(u_{i+1} - u_i) - k(u_i - u_{i-1}) = k(u_{i+1} + u_{i-1} - 2u_i) \tag{4}$$

[1] The requirement $|\mathbf{v}| = 1$ can be dropped if one considers the normalized quadratic form $Q = \mathbf{v}^T \cdot \mathsf{K} \cdot \mathbf{v}/(\mathbf{v}^T \cdot \mathbf{v})$.

At rest the masses are separated by a distance Δx, and so the rest position of the i-th mass can be taken as $i\Delta x$. Now define a *continuous* function $u(x,t)$ such that the displacements are given by

$$u_i(t) = u(i\Delta x, t). \tag{5}$$

In essence, we have now labelled the i-th mass by its rest position $i\Delta x$. (This is called a material, or Lagrangian, label in the field of continuum mechanics, which includes fluid dynamics.) This means that $\ddot{u}_i = u_{tt}(i\Delta x, t)$, where the subscript denotes partial t-derivatives at fixed location x.

Now consider the continuum limit $N \to +\infty$ whilst $N\Delta x$ =const. such that the length of the chain remains constant. Assuming smoothness of u on scales $\propto \Delta x$ allows Taylor-expanding the right-hand side of (4) around $x = i\Delta x$, which gives (dropping time-dependence at the moment)

$$\begin{aligned}
u_i &= u(i\Delta x) && (6)\\
u_{i+1} &= u(i\Delta x + \Delta x) = u(i\Delta x) + \Delta x u_x(i\Delta x) + \frac{\Delta x^2}{2} u_{xx}(i\Delta x) + O(\Delta x^3) && (7)\\
u_{i-1} &= u(i\Delta x - \Delta x) = u(i\Delta x) - \Delta x u_x(i\Delta x) + \frac{\Delta x^2}{2} u_{xx}(i\Delta x) + O(\Delta x^3) && (8)
\end{aligned}$$

where subscripts denote partial x-derivatives at fixed time t. Substituting in (4) leads to cancellations and

$$m u_{tt} = k\Delta x^2\, u_{xx} + O(\Delta x^3). \tag{9}$$

As $\Delta x \to 0$ the remainder term becomes negligible in the limit. This equation holds for all i, and hence holds for all x in the limit.

From elementary physics the spring parameter k is inversely proportional to the spring rest length Δx, i.e. $k = s/\Delta x$ for some positive constant s with units of force. Furthermore, the individual mass m is related to the mass density per unit length μ (which has units of mass divided by length) by $\mu = m/\Delta x$. Therefore, (9) can be rewritten as

$$\mu \Delta x u_{tt} = s\Delta x\, u_{xx}, \tag{10}$$

or finally in the form

$$\boxed{u_{tt} - c^2 u_{xx} = 0} \tag{11}$$

where $c = \sqrt{s/\mu}$ and has units of speed. This is the famous *wave equation.*

A different physical derivation for transversal oscillations of a string arrives at the same equation (with $c = \sqrt{T/\mu}$ where T is the string tension), so mathematically those oscillations are equivalent.

2.3 Normal modes of a string

Consider (11) for the transversal vibration of a finite string with length L and homogeneous boundary conditions $u = 0$ at $x = 0$ and $x = L$. Seeking normal modes:

$$u = \hat{u}(x)\exp(-i\omega t) \tag{12}$$

leads to the harmonic oscillator equation (in x)

$$\frac{\mathrm{d}^2\hat{u}}{\mathrm{d}x^2} + \frac{\omega^2}{c^2}\,\hat{u} = 0 \tag{13}$$

with general solution

$$\hat{u}(x) = A\cos(kx) + B\sin(kx), \quad k = \omega/c. \tag{14}$$

The boundary condition at $x = 0$ implies $A = 0$ and at $x = L$ we get

$$B\sin(kL) = 0. \tag{15}$$

Either $B = 0$ (trivial solution) or $kL = \pi, 2\pi, \ldots$. Therefore, we obtain an infinite number of normal modes indexed by a *mode number* $n = 1, 2, 3, \ldots$ such that $k = n\pi/L$:

$$\begin{array}{r|c|c|c} n = & 1 & 2 & 3 \\ k = & \frac{\pi}{L} & \frac{2\pi}{L} & \frac{3\pi}{L} \\ \omega = & \frac{\pi c}{L} & \frac{2\pi c}{L} & \frac{3\pi c}{L} \end{array}$$

The spatial normal modes $\sin(n\pi x/L)$ are linearly independent, indeed they are orthogonal vectors in Hilbert space:

$$\int_0^L \sin(m\pi x/L)\sin(n\pi x/L)\,\mathrm{d}x = \frac{L}{2}\delta_{mn}. \tag{16}$$

Adding negative integers n would not add linearly independent modes, because $n \to -n$ simply flips the sign of $\sin(n\pi x/L)$.

The most general real-valued normal mode belonging to a given mode number n is

$$u(x,t) = [a_n\cos(\omega_n t) + b_n\sin(\omega_n t)]\sin(k_n x) \tag{17}$$

with $k_n = n\pi/L$ and $\omega_n = ck_n$. Superposition of all possible modes gives the general solution as an infinite sequence

$$u(x,t) = \sum_{n=1}^{+\infty}[a_n\cos(\omega_n t) + b_n\sin(\omega_n t)]\sin(k_n x). \tag{18}$$

The coefficients are determined from the initial conditions via

$$u(x,0) = \sum_{n=1}^{+\infty} a_n\sin(k_n x) \tag{19}$$

$$u_t(x,0) = \sum_{n=1}^{+\infty} b_n\omega_n\sin(k_n x). \tag{20}$$

These are Fourier sine series, which have inversion formulas (cf. (16))

$$a_n = \frac{2}{L}\int_0^L u(x,0)\sin(k_n x)\,\mathrm{d}x \tag{21}$$

$$b_n = \frac{2}{\omega_n L}\int_0^L u_t(x,0)\sin(k_n x)\,\mathrm{d}x. \tag{22}$$