Senior Honors Class: Mathematical Wave Dynamics

Spring 2004 — Oliver Bühler & Alexander Barnett

Summary notes for lecture 3. February 3, 2004.

3.1 Traveling solutions

The evolution in time of the displacement $u(x,t)$ of a string is governed by the 1D (one-dimensional) wave equation,

$$u_{tt} - c^2 u_{xx} = 0 \qquad \text{(WE)}.$$

We consider *free space* (no boundaries), and look for solutions which travel at constant velocity v but otherwise remain unchanged. Such solutions have the form

$$u(x,t) = f(x - vt) \equiv f(z), \qquad \text{where } z(x,t) \equiv x - vt.$$

On a *spacetime* plot where x is horizontal axis and t vertical, contours of z are parallel lines with $1/v$ as their slope. f only depends on z. After a time t the original function has moved a distance vt, since $u(x,t) = u(x - vt, 0)$.
We take some derivatives using $z_x = 1$, $z_t = -v$ and the chain rule:

$$\begin{aligned}
f_x &= \frac{df}{dz} z_x = f' \qquad \text{with the notation } f' \equiv \frac{df}{dz} \\
f_t &= f' z_t = -v f' \\
f_{xx} &= \frac{\partial}{\partial x} f_x = \frac{d}{dz}(f') z_x = f'' \\
f_{tt} &= -v \frac{\partial}{\partial x} f_x = v^2 f''
\end{aligned}$$

Substituting into WE gives $v^2 f'' - c^2 f'' = 0$ which can hold for nontrivial $f(z)$ iff $v = \pm c$. Thus *any* function f can travel at speed c, either to the left or to the right. The general solution is a sum of general right- and left-going solutions,

$$\boxed{u(x,t) \;=\; f(x - ct) \;+\; g(x + vt), \qquad \text{for any functions } f, g.} \tag{1}$$

The two spacetime lines of influence $x \pm ct = \text{const}$ are called *characteristics* of the WE.
Is Eq.(1) complete? If it can represent any Initial Value Problem (IVP),

$$u(x,0) = U(x) \tag{2}$$
$$u_t(x,0) = V(x), \tag{3}$$

then it is complete. Inserting Eq.(1 at $t = 0$ into the above gives

$$f(x) + g(x) = U(x), \tag{4}$$
$$-c f_x + c g_x = V(x).$$

Integrating the latter over the domain $x' \in [-\infty, x]$, and assuming $f, g \to 0$ as $x \to -\infty$ gives

$$-f(x) + g(x) = \frac{1}{c} \int_{-\infty}^{x} V(x') dx'. \tag{5}$$

Note that since x is used as a limit, we needed to introduce a new integration variable x'. Eqs.(4) and (5) can now be added and subtracted to give

$$
\begin{aligned}
f(x) &= \tfrac{1}{2}U(x) - \frac{1}{2c}\int_{-\infty}^{x} V(x')dx' \\
g(x) &= \tfrac{1}{2}U(x) + \frac{1}{2c}\int_{-\infty}^{x} V(x')dx'.
\end{aligned}
$$

Hence we can match to any (sensible) initial conditions. If $V(x) = 0$ (released from rest), $f = g = \frac{1}{2}U$, so half goes in each direction.

3.2 Reflection at a boundary

For waves in *region of interest* $x < 0$ we impose some boundary condition (BC) at $x = 0$. We send in right-going pulse $f(x - ct)$ from the left (this is our IVP). We use the **Method of images** by finding $u(x,t)$ such that

1. u obeys free-space WE,
2. u obeys the BC for all time t,
3. u corresponds to our IVP in the region of interest.

The trick is to add an *image* pulse outside the region of interest. The two main types of BC give:

BC	name	math name	image function	after reflection
$u(0,t) = 0$	fixed end	Dirichlet	$-f(-x-ct)$	pulse inverted
$u_x(0,t) = 0$	free end	Neumann	$f(-x-ct)$	pulse retains sign

3.3 Energy

3.3.1 SHO

Multiplying the SHO ODE $mx_{tt} + kx = 0$ by x_t gives

$$m x_t x_{tt} + k x x_t = 0$$

which can be recognized as an exact t-derivative,

$$E_t = 0, \qquad \text{where } E(t) \equiv \tfrac{1}{2}m(x_t)^2 + \tfrac{1}{2}kx^2.$$

E is total energy of system, sum of kinetic and potential energy. This states that in an SHO, $E = \text{const}$.

3.3.2 Waves in 1D with constant tension

We do something similar for 1D WE, $\mu u_t t - s u_x x = 0$. Consider $\mu = \mu(x)$ varies in space but s (tension) is constant. Recall wave speed $c(x) = \sqrt{s/\mu(x)}$. Multiply WE by u_t:

$$\mu(x) u_t u_{tt} - s u_{xx} u_t = 0$$

and using $(u_t^2)_t = 2u_t u_{tt}$, $(u_x^2)_t = 2u_x u_{xt}$ and $(u_x u_t)_x = u_x u_{xt} + u_{xx} u_t$ this gives

$$E_t + F_x = 0 \tag{6}$$

where $E(x,t) \equiv \frac{1}{2}\mu(x)u_t^2 + \frac{1}{2}su_x^2$ is an *energy density* and $F(x,t) \equiv -su_x u_t$ is a *flux.*
Any equation of the form Eq.(6) is a *conservation law* since integrating it over domain $x \in [a, b]$ gives

$$\frac{\partial}{\partial t}\int_a^b E(x,t)dx \;=\; F(a,t) - F(b,t).$$

Rate of change in total energy inside the region equals flux (power) entering at a minus flux leaving at b. In other words, the total amount of E is conserved: changes can only be accounted for by flux at the boundary.
For $a = -\infty$, $b = +\infty$ and $u \to 0$ at $x = \pm\infty$, we have $\int_a^b E dx = \text{const}$.
We can show using the two above types of BC that flux $F = 0$ at each BC, so that no energy can pass from inside to outside of the region of interest. In other words, reflection of energy is 100% efficient.

3.4 Plane waves

Normal modes of WE in free space at single freq $\omega > 0$.

$$e^{ikx}e^{-i\omega t} = e^{ikx - i\omega t} = f(x - ct)$$

since $k = \omega/c$, so this (complex) wave travels to the right. $e^{-ikx-i\omega t}$ travels to the left. Wavelength $\lambda = \pi/k$ with k the *wavenumber*. General *real* solution is

$$\boxed{u_\omega(x,t) \;=\; \text{Re}\left[ae^{ikx-i\omega t} + be^{-ikx-i\omega t}\right]}$$

Taking the real part is essential (or adding the complex conjugate can similarly be used).
We make a **standing wave** from counter-propagating plane waves of equal magnitude, *e.g.* for $b = -a = -1$:

$$u_\omega(x,t) \propto (e^{ikx} - e^{-ikx})(e^{-i\omega t} + e^{i\omega t}) = 4\sin(kx)\cos(\omega t).$$

This standing wave is a checkerboard in spacetime. It has nodes where $u = 0$ for all time. This solution is identical to the string normal modes from lecture 2, when the string length L corresponds to $n\lambda/2$, the distance between nodes ($n \in \mathbb{Z}$).

3.5 Material interface

Scattering theory: send in plane wave of amplitude 1, what comes out (reflected and transmitted)?
Material has $s =$ const everywhere, but density μ for $x < 0$ and μ' for $x > 0$. Wave speed is therefore c in $x < 0$ and c' in $x > 0$.
Assume time evolution factor $e^{-i\omega t}$ for everything, and that real part will be taken to get physical quantities. Then we can do eveything with plane waves in x only:

$$\text{General solution:} \qquad u(x) = \begin{cases} e^{ikx} + ae^{-ikx} & x < 0 \\ be^{ik'x} & x > 0 \end{cases}$$

Note a is the *amplitude reflection coefficient*, b is the *amplitude transmission coefficient*. Also note $\omega = ck = c'k'$. Continuity of u at $x = 0$, and continuity of slope u_x at $x = 0$ (which follows from considering equal tension s on both sides) gives

$$1 + a = b \tag{7}$$
$$ik - ika = ik'b \tag{8}$$

Solving these two for a, b gives

$$a = \frac{c' - c}{c' + c}, \qquad b = \frac{2c'}{c' + c}.$$

Light rope going to heavy rope ($c' \ll c$): $a \approx -1$, like Dirichlet BC.
Heavy rope going to light rope ($c' \gg c$): $a \approx +1$, like Neumann BC.

Project idea: solve scattering numerically for more complicated 1D variable-material systems, resonators, filters, etc.

Energy conservation at material interface: using above answer we can check

$$k|1|^2 = k|a|^2 + k'|b|^2$$

that is, total power in equals total power out. We will need the result that time-averaged power carried by a plane wave $ae^{\pm ikx}$ is $\overline{F} = \overline{-su_x u_t} = 2\omega sk|a|^2$. This calculation will be done next time.

Impedance: if you want to go further into the above material interface, and consider variable s too, the best way is via *impedance* $z = \sqrt{s\mu} = \mu c = s/c$. Impedance is ratio of force to velocity, and reflection and transmission coefficients are best expressed in terms of impedance ratios. (Equal impedances implies full transmission). See waves reference books for more.