Senior Honors Class
Spring 2004 — Oliver Bühler & Alexander Barnett
Summary notes for lecture 4. 10 Feb 2004

4.1 Refraction in one dimension

Consider variable wave speed $c(x)$. Define **index of refraction** $n(x)$ such that $c = c_0/n(x)$, where c_0 is a constant reference wave speed (e.g. the wave speed at some position x_0). Wave equation becomes

$$n^2 u_{tt} - c_0^2 u_{xx} = 0. \tag{1}$$

Assume normal mode in time $u = \hat{u}(x)\exp(-i\omega t)$, this gives ODE for $\hat{u}(x)$

$$\hat{u}_{xx} + k^2(x)\hat{u} = 0 \quad \text{where} \quad k^2(x) \equiv \omega^2 n^2(x)/c_0^2. \tag{2}$$

This is the simple harmonic oscillator equation with varying frequency.

Assume $n(x)$ varies little over one wavelength $2\pi/k$. Formally, if $n(x)$ changes significantly[1] over a length L then this implies $kL \gg 1$.

Make **slowly varying wavetrain** Ansatz for $\hat{u}(x)$:

$$\hat{u}(x) = A(x)\exp(i\phi(x)), \tag{3}$$

where $\phi(x)$ is the spatial **phase** of the wave. For a plane wave $\phi = kx$. Demand that the slowly varying wavetrain looks locally like a plane wave with the correct wavenumber:

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} = k(x) = \frac{\omega n(x)}{c_0} \quad \Rightarrow \quad \phi = \phi_0 + \int_{x_0}^{x} k(\bar{x})\mathrm{d}\bar{x}, \tag{4}$$

where $k(x)$ is a root from (2) above.

The wave amplitude can be found from the energy law:

$$u_t n^2(x) u_{tt} - c_0^2 u_t u_{xx} = 0 \tag{5}$$

$$\left[n^2(x)\frac{u_t^2}{2}\right]_t - c_0^2(u_t u_x)_x + c_0^2 u_{xt} u_x = 0 \tag{6}$$

$$\left[n^2(x)\frac{u_t^2}{2} + c_0^2\frac{u_x^2}{2}\right]_t + \left\{-c_0^2 u_t u_x\right\}_x = 0. \tag{7}$$

The quantity in square brackets is the wave energy $E(x,t)$ and the quantity in curly brackets is the energy flux $F(x,t)$. So get the conservation law

$$E_t + F_x = 0. \tag{8}$$

Define the time average over one wave period $T = 2\pi/\omega$ as

$$\overline{X} = \frac{1}{T}\int_t^{t+T} X(\bar{t})\mathrm{d}\bar{t}. \tag{9}$$

[1]For instance, if $n = a + b\tanh(x/L)$ then n changes from $a-b$ to $a+b$ over a length $\approx L$ centred at the origin (plot this function).

For periodic functions with period T this average does not depend on t. Clearly, $\overline{u} = 0$. The average of the wave energy $\overline{E}(x)$ is not zero, but it does not depend on time anymore. Hence, the average of (8) reduces to $(\overline{F})_x = 0$, i.e. the mean flux is a constant:

$$\overline{F} = -c_0^2\,\overline{u_t u_x} = \text{const.} \tag{10}$$

To evaluate such a quadratic average is easy for a plane wave. For a slowly varying wave-train we can evaluate it approximately by using the plane-wave result with the appropriate local wavenumber $k(x)$.

The general plane-wave result is that if

$$u = \Re A \exp(i(kx - \omega t)) \tag{11}$$

$$v = \Re B \exp(i(kx - \omega t)) \tag{12}$$

then

$$\overline{uv} = \frac{1}{2}\Re(A^* B) = \frac{1}{2}\Re(B^* A), \tag{13}$$

where A^* is the complex conjugate of the complex number A.

Therefore

$$\overline{u_t u_x} = \frac{1}{2}\Re((-i\omega A)^*(ikA)) = \frac{1}{2}\Re((i\omega A^*)(ikA)) = \frac{-1}{2}k\omega|A|^2. \tag{14}$$

The constancy of $\overline{F}$ then implies that $k|A|^2 = $const, or

$$\frac{A(x)}{A_0} = \sqrt{\frac{k_0}{k(x)}} \tag{15}$$

where A_0, k_0 occur at some x_0.

Together with the phase solution above, we arrive at the so-called "WKB" solution to (2):

$$\boxed{u(x,t) = u_0\sqrt{\frac{k_0}{k(x)}}\exp\left(i\int_{x_0}^{x} k(\bar{x})\mathrm{d}\bar{x}\right)\exp(-i\omega t)}. \tag{16}$$

Here $u_0 = u(x_0, 0)$ and $k_0 = \omega/c_0$ such that $k(x) = n(x)k_0$.

Increased index of refraction n implies decreased speed c, increased wavenumber k, decreased wavelength $2\pi/k$, and decreased wave amplitude $A \propto k^{-1/2}$. For example, $n = \exp(\alpha x)$ is increasing with x for $\alpha > 0$. The WKB solution with $x_0 = 0$ is

$$u(x,t) = u_0 \exp\left(-\frac{\alpha x}{2}\right)\exp\left(i\frac{k_0}{\alpha}[\exp(\alpha x) - 1]\right)\exp(-i\omega t). \tag{17}$$

This shows exponential amplitude decay and exponentially growing phase.

Summary: (1) slowly varying wavetrain looks locally like a plane wave with wavenumber $k(x)$ such that $\omega = c(x)k(x)$; wavenumber defined via phase $\phi(x)$. (2) amplitude from energy flux constancy; note that energy density is not constant. Indeed, show that for a plane wave we have

$$\overline{F} = c(x)\overline{E}, \tag{18}$$

so $\overline{E} \propto n$ at constant $\overline{F}$. (1) is the topic of **geometric** wave theory, and (2) is topic of **physical** wave theory.

4.2 Two-dimensional waves

The two-dimensional wave equation for $u(x, y, t)$ is

$$u_{tt} - c^2(u_{xx} + u_{yy}) = 0 \tag{19}$$

and similarly in higher dimensions. A two-dimensional plane wave has the form

$$u = A\exp(i[kx + ly - \omega t]) = A\exp(i[\boldsymbol{k} \cdot \boldsymbol{x} - \omega t]), \tag{20}$$

where the wavenumber vector $\boldsymbol{k}$ has components $\boldsymbol{k} = (k, l)$ and the position vector $\boldsymbol{x} = (x, y)$. The lines of constant spatial phase $\phi = kx + ly$ are straight lines with slope $-k/l$; such lines are called **phase lines** (or phase hyper-surfaces in higher dimensions). At fixed time t, a wave crest or trough can be identified with a particular value of the phase ϕ. The wavenumber vector $\boldsymbol{k}$ is perpendicular to the phase lines and measures the phase increase per unit length in that direction. The wavelength of the wave measured along $\boldsymbol{k}$ is $2\pi/\kappa$, where

$$\kappa = \sqrt{k^2 + l^2} \tag{21}$$

is the wavenumber vector magnitude.

The full wave phase is defined as

$$\theta(x, y, t) = kx + ly - \omega t = \phi - \omega t. \tag{22}$$

Consider the phase θ at a point (x, y) at time t. If time increases by $\mathrm{d}t$ and (x, y) by $(\mathrm{d}x, \mathrm{d}y)$, then the phase changes by its total differential

$$\mathrm{d}\theta = k\mathrm{d}x + l\mathrm{d}y - \omega\mathrm{d}t. \tag{23}$$

In order to stay with a particular wave crest, say, we require the phase θ to remain constant: $\mathrm{d}\theta = 0$. This gives

$$k\frac{\mathrm{d}x}{\mathrm{d}t} + l\frac{\mathrm{d}y}{\mathrm{d}t} = \omega. \tag{24}$$

This has multiple solutions; two are most relevant. The **scalar** phase speed follows from keeping $y =$const.:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\omega}{k}, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = 0. \tag{25}$$

It is useful at boundaries where $y =$const. The **vectorial** phase speed follows from demanding that the phase speed is parallel to $\boldsymbol{k}$, i.e. that it is normal to the phase lines. This gives

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\omega}{\kappa^2}k, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\omega}{\kappa^2}l. \tag{26}$$

This phase speed is so important that we use the notation

$$\boldsymbol{u}_p = \frac{\omega}{\kappa^2}\boldsymbol{k} \tag{27}$$

for it. Notice that $\boldsymbol{u}_p$ would have the opposite sign if we had started with $\exp(\ldots + i\omega t)$ in the plane wave definition.

A plane wave solves the wave equation if

$$\omega^2 = c^2(k^2 + l^2) = c^2\kappa^2. \tag{28}$$

For square or circular domain shapes one can derive a full set of normal modes that are zero at the boundary. However, no simple general theory exists for arbitrary domain shapes.

> **Project suggestion:** Investigate Kac's question (1966): "Can you hear the shape of a drum?". References: www.ams.org/new-in-math/cover/199706.html, and paper in www.maths.ox.ac.uk/∼chapman. Or look at related questions such as "Can you hear the shape of a lake?"

4.3 One-dimensional refraction in two dimensions

Let the refractive index by a function of x only: $n(x)$. This is almost the same as one-dimensional refraction. The wave equation becomes

$$n^2(x)u_{tt} - c_0^2(u_{xx} + u_{yy}) = 0 \tag{29}$$

and assuming normal modes in t and y means

$$u = \hat{u}(x)\exp(i[ly - \omega t]). \tag{30}$$

This gives (2) again, this time with

$$k^2 = \frac{\omega^2 n^2(x)}{c_0^2} - l^2. \tag{31}$$

The WKB solution follows as before, provided k^2 does not become negative. This can happen if n decreases below the critical value

$$n = lc_0/\omega \tag{32}$$

at which $k^2 = 0$. Waves cannot propagate into a region where $k^2 < 0$.

> **Project suggestion:** Investigate wave refraction in a practical setting such as water waves coming into a beach.