Senior Honors Class

Spring 2004 — Oliver Bühler & Alexander Barnett

Summary notes for lecture 5. 17 Feb 2004

5.1 Dispersion (cf. Feynman lectures I, 48ff)

One-way wave equation, advection equation (non-dispersive):

$$u_t + cu_x = 0, \quad c > 0 \tag{1}$$

solved by $u = f(x - ct)$ for any function f. Right-going wave only. General initial-value problem with $u(x, 0) = u_0(x)$ therefore has the simple solution $u(x, t) = u_0(x - ct)$, i.e. the initial function moves to the right with speed c. There is no change of shape. The plane wave solution is

$$u = A \exp(i(kx - \omega t)) \quad \Rightarrow \quad \omega = ck. \tag{2}$$

So phase speed is $u_p = \omega/k = c$ for all k.

Dispersive linear KdV equation:

$$u_t - \beta u_{xxx} = 0, \quad \beta > 0. \tag{3}$$

A plane wave Ansatz now leads to the **dispersion relation**

$$\omega = \beta k^3, \quad \Rightarrow \quad u_p = \beta k^2. \tag{4}$$

So the phase speed is always positive, but depends on k; this is called **dispersion**. Shorter waves travel faster. Initial-value problem simple only for normal modes:

$$u_0 = \exp(ikx) \quad \Rightarrow \quad u(x, t) = \exp(ik(x - u_p t)). \tag{5}$$

General solution possible by superposition of normal modes (Fourier series/transform).

5.2 Amplitude modulation: beats

Two normal modes with different wavenumbers k_1 and k_2:

$$u_0 = \exp(ik_1 x) + \exp(ik_2 x) = 2\cos\left(\frac{k_1 - k_2}{2} x\right) \exp(\frac{i}{2}(k_1 + k_2)x). \tag{6}$$

This is a wave with wavenumber $0.5(k_1 + k_2)$ modulated in amplitude by the cosine factor. Inspection shows that neighbouring cosine amplitude peaks occur at distances $2\pi/(k_1 - k_2)$.

At later times

$$\begin{aligned} u(x, t) &= \exp(i(k_1 x - \omega_1 t)) + \exp(i(k_2 x - \omega_2 t)) & (7) \\ &= 2\cos\left(\frac{k_1 - k_2}{2} x - \frac{\omega_1 - \omega_2}{2} t\right) \exp(\frac{i}{2}((k_1 + k_2)x - (\omega_1 + \omega_2)t)). & (8) \end{aligned}$$

Hence the phase of the modulated wave moves with speed

$$u_p = \frac{\omega_1 + \omega_2}{k_1 + k_2} \tag{9}$$

whilst the amplitude envelope moves with speed

$$u_{AM} = \frac{\omega_1 - \omega_2}{k_1 - k_2}. \tag{10}$$

These are not the same speeds unless $\omega = ck$, i.e. unless the system is non-dispersive. These formulas hold for arbitrary wavenumbers, but most important is the limit $k_1 \to k_2 = k$, which gives the amplitude modulation of a primary wave k by a neighbouring wavenumber $k + \mathrm{d}k$. We then obtain

$$u_p(k) = \frac{\omega(k)}{k} \quad \text{and} \quad u_{AM}(k) = \frac{\mathrm{d}\omega(k)}{\mathrm{d}k}. \tag{11}$$

Therefore amplitude modulations move at a different speed than the phase speed if the wave is dispersive. In our example $u_{AM} = 3\beta k^2 = 3u_p$, so the amplitude moves three times as fast as the phase.

5.3 Frequency modulation

Consider a slowly varying wavetrain $u = A\exp(i\theta)$ where the wavenumber and frequency are related to the phase $\theta(x,t)$ by

$$k(x,t) \equiv \theta_x, \quad \omega(x,t) \equiv -\theta_t \tag{12}$$

Partial derivatives commute and hence $\theta_{xt} = \theta_{tx}$ implies

$$k_t + \omega_x = 0, \tag{13}$$

which is a conservation law for spatial wave phase difference with density k and flux ω. For a slowly varying wavetrain we *demand* that k and ω also satisfy the dispersion relation for plane waves $\omega = \beta k^3$. This turns (13) into a single equation for $k(x,t)$:

$$k_t = -\frac{\mathrm{d}\omega}{\mathrm{d}k}\, k_x = -3\beta k^2\, k_x. \tag{14}$$

From calculus we know that the constant-value-contours of any function $k(x,t)$ have slope

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)_{k=\text{const.}} = -\frac{k_t}{k_x} = 3\beta k^2. \tag{15}$$

This is the speed at which individual values of k travel. This shows that frequency (or wavenumber) modulations travel at the same speed as amplitude modulations:

$$u_{FM}(k) = u_{AM}(k). \tag{16}$$

(Note that faster waves can catch up with slower waves, in which case the assumption of a single slowly varying wavetrain needs to be amended. However if faster waves are initially to the right of slower waves then this will not happen.)

5.4 Energy flux

The advection equation (1) and the linear KdV equation (3) both have quadratic conservation laws associated with them. They are found by multiplying the equations by u and rearranging them. As a rule of thumb: if the highest time derivative is of order n, then multiply the equation by the $(n-1)$th time derivative to get the energy law. The results are

$$uu_t + cuu_x = 0 \tag{17}$$

$$\left(\frac{u^2}{2}\right)_t + \left(c\frac{u^2}{2}\right)_x = 0 \tag{18}$$

and

$$uu_t - \beta uu_{xxx} = 0 \tag{19}$$

$$\left(\frac{u^2}{2}\right)_t + \left(\beta\frac{u_x^2}{2} - \beta uu_{xx}\right)_x = 0. \tag{20}$$

In the non-dispersive system we clearly have $F = cE$ at for all shapes of u, and hence energy always flows with speed c in that system. This is not the case in the dispersive system, where the ratio of flux to energy density depends on the shape of u. For a plane wave (2), however, a simple relationship between the time-averaged flux $\overline{F}$ and the time-averaged energy density $\overline{E}$ is easily found to be

$$\overline{F} = u_{EF}\,\overline{E}, \quad \text{with} \quad u_{EF} = 3\beta k^2. \tag{21}$$

Therefore, in a plane wave the energy flows with speed

$$u_{EF} = u_{FM} = u_{AM}. \tag{22}$$

We have now found the same answer to three different questions: what is the speed of amplitude modulations, of frequency modulations, and of plane-wave energy flow? It is time to give this speed its proper name, which is the celebrated **group** velocity

$$\boxed{u_g \equiv \frac{\mathrm{d}\omega}{\mathrm{d}k}}. \tag{23}$$

Another (equivalent) definition of dispersion is that for dispersive waves $u_g \neq u_p$, i.e. wave crests travel at a different speed than wavetrain modulations.

5.5 Wavepackets

A single wavepacket is an initial condition of the form

$$u_0(x) = A(x)\exp(ikx) \tag{24}$$

where the amplitude envelope A varies slowly compared to the wavelength $2\pi/k$ and is assumed to vary from zero at $x \to \pm\infty$ to a maximum at the origin. For example, a Gaussian envelope has the shape

$$A = \exp\left(-\frac{x^2}{L^2}\right) \tag{25}$$

where the envelope scale L satisfies $kL \gg 1$. At later times the approximate solution to a dispersive wave equation with this initial condition will then be given by

$$\boxed{u(x,t) = A([x - u_g(k)t]) \exp(ik[x - u_p(k)t])}, \tag{26}$$

where $u_p = \omega(k)/k$ and u_g follows from (23). The accuracy of this wavepacket approximation depends on the size of kL and on the integration time t, for longer times this becomes less and less accurate.

5.6 Localized initial conditions (cf. Whitham "Linear & nonlinear waves, 11ff)

A localized initial condition would be something like $u_0 = A$ from (25). What happens in a dispersive system is that such a bump will emit a slowly varying wavetrain with variable $k(x,t)$. At fixed time t faster waves will be found further away from the origin. Specifically, at time t a particular wavenumber k will be found at a location $x = tu_g(k)$. Inverting this relation we can find the function $k(x,t)$:

$$u_g(k) = \frac{x}{t} \quad \Rightarrow \quad 3\beta k^2 = \frac{x}{t} \quad \Rightarrow \quad k = \pm\sqrt{\frac{x}{3\beta t}}. \tag{27}$$

This only works for $x > 0$ and the sign of k does not matter. We see that for fixed x the local wavenumber decreases $\propto 1/\sqrt{t}$, whilst for fixed t the local wavenumber increases $\propto \sqrt{x}$. For fixed x/t the wavenumber is constant. The last fact is obvious because moving at fixed x/t means moving with fixed group velocity. In general, we call the path traced out by the group velocity a group velocity **ray**.

5.7 Ray tracing

For a single wavepacket the trajectory of the wavepacket is a group velocity ray. We can consider the trajectory as given by $x(t)$ with some initial condition $x(0) = x_0$. Then we have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u_g. \tag{28}$$

The group velocity depends on k and following the wavepacket we can consider $k(x,t)$ on the group velocity ray $x(t)$. This means that on this ray k depends only on time, with some initial condition $k(0) = k_0$. The "evolution" equation for k is of course

$$\frac{\mathrm{d}k}{\mathrm{d}t} = 0 \quad \Rightarrow k(t) = k_0. \tag{29}$$

Together these two ODEs are called the **ray-tracing equations**. Because k is constant $x(t) = x_0 + tu_g(k_0)$ is the simple solution.

Now consider refraction of the wavepacket by a slowly varying $\beta(x)$. Returning to (12) and (13) we can define a frequency function

$$\Omega(k,x) = \beta(x)k^3 \tag{30}$$

such that $\omega = \Omega$ along the ray. Note carefully that Ω is viewed as a function of two different independent variables k and x. For instance the local group velocity is given by $u_g = \partial\Omega/\partial k = 3\beta(x)k^2$, where x is kept constant in the partial derivative. Substitution in (12) and the chain rule gives

$$k_t + u_g k_x = -\frac{\partial \Omega}{\partial x} = -\beta'(x)k^3. \tag{31}$$

The left-hand side is precisely the time derivative following the wavepacket (i.e. along the group velocity ray). Therefore the generalization of (28) and (29) to refraction by a slowly varying background state $\beta(x)$ is given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = +\frac{\partial \Omega}{\partial k} = 3\beta(x)k^2 \tag{32}$$

$$\frac{\mathrm{d}k}{\mathrm{d}t} = -\frac{\partial \Omega}{\partial x} = -\beta'(x)k^3 \tag{33}$$

Now both x and k vary along the ray. For instance, if $k_0 > 0$ and β increases with x then k decreases following the wavepacket trajectory. The change of ω along the ray is easily computed as well:

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\partial \Omega}{\partial k}\frac{\mathrm{d}k}{\mathrm{d}t} + \frac{\partial \Omega}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} \tag{34}$$

$$= -\frac{\partial \Omega}{\partial k}\frac{\partial \Omega}{\partial x} + \frac{\partial \Omega}{\partial k}\frac{\partial \Omega}{\partial x} = 0. \tag{35}$$

This holds in general and tells us that ω does not change, i.e. the wavepacket always exhibits the same frequency. This implies that along a ray

$$u_g = 3\beta k^2 = 3\omega/k \propto 1/k. \tag{36}$$

Hence if k decreases then the wave speeds up! This would have been hard to guess.

The above would change if the background were time-dependent as well: i.e. $\beta(x, t)$ such that we have $\Omega(k, x, t)$. Then one can easily show that

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\partial \Omega}{\partial t}. \tag{37}$$

Note that this equation for ω along the ray is already implied by the system of ODEs (32).

The system of coupled ODEs (32) is an example of a canonical Hamiltonian system for the conjugate pair of variables (k, x) with Hamiltonian function $\Omega(k, x)$, which is given by the dispersion relation. Such a system is very easy to integrate on a computer and allows investigating the behaviour of wavepackets dictated by the dispersion relation.

Project suggestions: Investigate dispersive wave in quantum mechanics (particle waves) or for water waves, which are dispersive in deep water but non-dispersive in shallow water.