

Senior Honors Class: Mathematical Wave Dynamics

Spring 2004 — Oliver Bühler & Alexander Barnett

Summary notes for lecture 6. February 24, 2004.

6 Nonlinear waves

6.1 Traffic flow (cf. Billingham & King, §7.1)

The x -axis is our one-dimensional (single-lane) ‘road’. Continuum approximation: $\rho(x, t)$ is local car density (cars per unit length). Integral form of the *law of conservation of cars*¹,

$$\frac{d}{dt} \int_a^b \rho(x, t) dx = F(a, t) - F(b, t) \quad (1)$$

holds for any fixed locations a, b , where $F(x, t)$ is local flux of cars (number of cars per unit time passing a fixed point). By taking $\lim a \rightarrow b$ this is equivalent to

$$\rho_t + F_x = 0, \quad (2)$$

its differential form (cf. Section 3.3.2). We also have flux $F = \rho v$ where v is local car velocity. A simple driver behavior model assumes that $v = v(\rho)$, drivers drive at max safe speed, which is a *decreasing* function of the local density. $v(\rho_{\max}) = 0$ where ρ_{\max} is the ‘bumper-to-bumper’ maximum density when traffic stops. Flux $F(\rho) = \rho v(\rho)$ is then unimodal function vanishing at 0 and ρ_{\max} , with peak at $\rho = \rho^*$ somewhere inbetween. Writing via chain rule $F_x = F_\rho \rho_x$, and defining $F_\rho(\rho) \equiv c(\rho)$ turns Eq. 2 into

$\rho_t + c(\rho)\rho_x = 0$	Traffic Equation (TE).
------------------------------	------------------------

 (3)

Nonlinearity is evident in the second term: it is generally not a linear function of ρ .

6.1.1 Nonexistence of traveling solutions

Inserting $\rho(x, t) = f(x - c_0 t)$ into TE gives $-c_0 f' + c(f) f' = 0$, that is, either

1. $f = \text{constant}$, is ‘boring’ but we will linearize about it in next section, or
2. $c(\rho) = c_0$, giving the familiar one-way wave equation $\rho_t + c_0 \rho_x = 0$ from Section 5.1.

There is no IVP which gives a non-constant traveling solution when $c(\rho)$ is non-constant.

6.2 Linearization about constant density

We write $\rho(x, t) = \rho_0 + \delta\rho(x, t)$ with $\delta\rho \ll \rho_0$, and since $c(\rho)$ is a smooth function of ρ , we can assume $c(\rho_0 + \delta\rho(x, t)) \approx c(\rho_0) \equiv c_0$ for sufficiently small $\delta\rho$. So the TE becomes,

$$\delta\rho_t + c_0 \delta\rho_x = 0,$$

just the linear 1-way WE whose const velocity c_0 is the slope of the $F(\rho)$ graph at the background density ρ_0 . Two cases,

¹NB you will not find this in your physics book, rather it expresses our simple model that no cars enter or leave the road

1. $\rho < \rho^*$: $c_0 > 0$ so characteristics point to the NW on spacetime plot.
2. $\rho > \rho^*$: $c_0 < 0$ so characteristics point to the NE on spacetime plot. Note cars still move forward ($v > 0$) even though $c_0 < 0$.

Note *Kinematic wave speed* c_0 is the velocity at which small variations in density, not cars themselves, are transported.

6.3 Full nonlinear solution

We want a *characteristic* line $X(t)$ which tells us how ρ is transported, *i.e.* $\rho = \text{const}$ along such a line. Writing this then using chain rule gives

$$\begin{aligned}\frac{d}{dt}\rho(X(t), t) &= 0, \\ \rho_t + X_t\rho_x &= 0.\end{aligned}$$

Comparing to TE shows characteristic has local velocity $X_t = c(\rho)$. But density is constant along a characteristic, so X_t , a function only of local density, must be too! The *characteristics are straight lines* even though the PDE is nonlinear. Solution $\rho(x, t)$ to IVP $\rho(x, 0) = \rho_0(x)$ is given *implicitly* by

$$\begin{aligned}\rho(X(t), t) &= \rho_0(x_0) && \text{(density preserved along characteristics)} \\ X(t) &= x_0 + c(\rho_0)t && \text{(characteristics are straight)}.\end{aligned}$$

In other words, given a spacetime point (x, t) , one needs to search for a characteristic $X(t)$ intersecting this point, and $\rho(x, t)$ will then be given by ρ at x_0 , the ‘launch point’ of this characteristic. Diagrams are helpful (see Billingham & King). A more compact expression of the above is

$$\rho = \rho_0(x - c(\rho)t),$$

an implicit nonlinear equation for ρ at the point (x, t) .

There are two classes of phenomenon,

1. ρ_0 decreasing with x , *i.e.* $c(\rho_0)$ increasing with x . Here, characteristics form *expansion fan* and the density smooths out with time,
2. ρ_0 increasing with x , *i.e.* $c(\rho_0)$ decreasing with x . Characteristics now point towards each other (density changes are made more steep with time), and when they cross a *shock wave* is formed. Here the PDE (TE) no longer useful (derivatives are infinite).

6.3.1 Time to first appearance of shock

By drawing straight lines it is easy to see that the earliest formation of a shock is due to adjacent (infinitesimally-separated) characteristics. Take their launch points to be x_1 and x_2 , and velocities c_1 and c_2 . Then linear motion gives $x_2 - x_1 = (c_1 - c_2)T$, which in the limit $x_2 \rightarrow x_1$ gives

$$T = -\frac{1}{dc_0/dx}$$

where $c_0(x) = c(\rho_0(x))$ is the initial velocity profile. Therefore the earliest shock forms at time

$$T_{\min} = \min_x \left[-\frac{1}{dc_0/dx} \right]. \quad (4)$$

This shows that unless $d\rho_0/dx \geq 0$ everywhere, shock formation is inevitable.

6.4 Inviscid Burger's Equation

We specialize to particular simple linear driver model $v(\rho) = v_0(1 - \rho/\rho_{\max})$, in which case $c(\rho) = v_0(1 - 2\rho/\rho_{\max})$ and $\rho^* = \rho_{\max}/2$. Map $c \leftrightarrow \rho$ is 1-1 invertible, so we can use $c(x, t)$ as our PDE variable, turning TE into

$c_t + cc_x = 0$	Inviscid Burger's Equation.
------------------	-----------------------------

(5)

Solutions to discontinuous IVP

$$c_0(x) = \begin{cases} c_L, & x < 0 \\ c_R, & x > 0 \end{cases} \quad (6)$$

fall in two classes.

6.4.1 $c_L < c_R$: Expansion fan solution

At future times the c values outside the fan remain at their initial values, but inside, the assumption that no shock persists allows us to use characteristics at intermediate velocities inside fan region. Giving,

$$c(x, t) = \begin{cases} c_L, & x < c_L t \\ x/t, & c_L t < x < c_R t \\ c_R, & x > c_R t \end{cases} \quad (7)$$

When corresponding densities are $\rho_L = \rho_{\max}$ and $\rho_R = 0$, this is known as the *green light problem* (think traffic). In this case kinematic waves propagate at $c_L = -v_0$ and $c_R = v_0$.

6.4.2 $c_L > c_R$: Moving shock solution

Eq. 4 tells us a shock forms immediately. What does it do then? We assume it moves along a trajectory $s(t)$, and that the solution $c(x, t)$ remains smooth for future times, everywhere in space apart from at $s(t)$. The PDE (TE) is useless at $s(t)$ since it gives infinities there, but we return to conservation law integral form. Writing $cc_x = (c^2/2)_x$ turns Eq. 5 into the conservation law

$$E_t + F_x = 0$$

with 'energy' $E = c$ and flux $F = c^2/2$. The integral form is therefore

$$\frac{d}{dt} \int_a^b E(x, t) dx = F(a, t) - F(b, t),$$

and we choose constants $a < s(t)$ and $b > s(t)$ for all t of interest. We split the integral into the sum of two parts either side of the shock,

$$\frac{d}{dt} \left(\int_a^{s_-(t)} E(x, t) dx + \int_{s_+(t)}^b E(x, t) dx \right) = F(a, t) - F(b, t),$$

where s_- (s_+) is infinitesimally to the left (right) of $s(t)$. Moving limits can be handled by a general rule from calculus,

$$\frac{d}{dt} \int_{p(t)}^{q(t)} g(x, t) dx = \int_{p(t)}^{q(t)} g_t(x, t) dx - p_t g(p(t), t) + q_t g(q(t), t).$$

Applying this to both integrals above gives

$$\left(\int_a^{s_-(t)} + \int_{s_+(t)}^b \right) E_t(x, t) dx - s_t \Delta E(t) = -\Delta F(t) \quad (8)$$

where we have defined the flux jump $\Delta F(t) \equiv F(s_+(t), t) - F(s_-(t), t)$ and the ‘energy’ jump $\Delta E(t) \equiv E(s_+(t)) - E(s_-(t))$. Since $E = c$ is constant to the left and right of $s(t)$, the integral above vanishes.² Rearranging then using our forms for E and F gives,

$$s_t = \frac{\Delta F(t)}{\Delta E(t)} = \frac{c_R^2/2 - c_L^2/2}{c_R - c_L} = \frac{c_R + c_L}{2}, \quad (9)$$

in other words *the shock moves at the average velocity on the two sides*. Again, if $\rho_L = 0$ and $\rho_R = \rho_{\max}$, this is known as the *red light problem*.

Note the nontrivial above result holds even for nonconstant velocity fields $c(x, t)$ on the two sides (the vanishing of the integral in Eq. 8 now relies on the argument that a, b can be taken arbitrarily close to $s(t)$ if arbitrarily short times are considered).

6.5 Viscosity

In the real world, drivers look ahead a short distance l and slow down to prevent a true shock from forming. The easiest continuum model is

$$c_t + cc_t - \nu c_{xx} = 0$$

where the new ‘viscous’ (so named for its role in fluid dynamics) or ‘diffusion’ (another name arising from its role in the heat equation) term, νc_{xx} smooths out the shock. A small ν gives a small l . The smoothed shock still evolves similarly to above.

6.6 Solitons

Contrast two equations we have studied.

1. Inviscid Burger’s Equation $u_t + uu_x = 0$: nonlinearity causes steepening of wavecrest, we have no traveling waves or normal modes, each part of the wave moves at its own ‘private’ speed, and shocks result.

²Billingham & King p.237–238 is disappointingly non-rigorous about this.

2. Linear KdV Equation $u_t - \beta u_{xxx} = 0$: linear, traveling wave normal modes do exist, dispersion causes widening of a localized wavepacket into component frequencies (slowly-varying wavetrain).

Can we find a situation where the nonlinear steepening and dispersive linear widening balance? If so, we may have localized traveling wave solutions. These are called *solitons*. We need both nonlinearity and dispersion, combining terms from both above equations,

$$u_t + \alpha u u_x + u_{xxx} = 0 \quad \text{KdV Equation.}$$

This is a common model for nonlinear dispersive waves, such as shallow water waves. They were first observed in shallow canals. Note the dispersive term corresponds to $\beta < 0$.

We demonstrate the soliton solution (following Schmidt). Substituting the familiar traveling form $u(x, t) = f(x - vt)$ into KdV gives

$$-vf' + \alpha f f' + f''' = 0,$$

then integrating gives

$$-vf + \frac{\alpha}{2} f^2 + f'' = C. \quad (10)$$

For a localized solution f and its derivatives vanish at $\pm\infty$ thus $C = 0$. Eq. 10 is the kinematic ODE for motion $f(x)$ with ‘time’ coordinate x of a unit-mass point particle under the nonlinear force law $F(f) = vf - \frac{1}{2}\alpha f^2$. This gives an intuitive grasp of traveling solution classes sharing velocity v . Near $f = 0$ the force approximates a *negative* spring constant (unstable, inverted SHO) and near $f = 2v/\alpha$ a positive one (stable SHO). A soliton corresponds to ‘release’ of this particle infinitesimally to the right of $f = 0$. What follows, as a function of ‘time’ x is: exponential growth, a violent flip over past the stable point and exponential decay back to $f = 0$. Such 1D kinematic problems can be solved analytically. It can be verified that

$$f(x) = \frac{3v}{\alpha} \cosh^{-2} \left(\frac{\sqrt{v}}{2} x \right) \quad \text{KdV soliton shape at velocity } v$$

is a solution to Eq. 10 which vanishes at $\pm\infty$. Notice that the amplitude $A = 3v/\alpha$ and width $\sim v^{-1/2}$ are both fixed once the soliton speed v is fixed. Thus solitons form a *one-parameter family*. The general rule is: faster = taller = narrower. Solitons are a multi-billion dollar industry in fiber-optic communications because they prevent dispersive distortion of modulated signals.

Note that small-amplitude oscillation about the stable point gives approximately sinusoidal solutions (so-called *cnoidal* traveling waves).

6.6.1 Soliton collisions

We note that solitons of different speeds interact with each other in a nonlinear fashion, but, remarkably, recover their original identities once they separate. The key difference between this and linear wave interactions is that the solitons suffer permanent position (phase) offsets relative to their initial trajectories. There are deep analytical results in this area of *soliton scattering theory*.

Project idea: Numerically simulate solitons, their collisions, etc, with applications in mind and parameter values corresponding to fiber-optics or water waves. For example, what is the effective ‘force’ between nearby solitons?