Chapter 1

Simple Lagrangian Model in 1 D

1.1 Ice Dynamics as Fluid Dynamics

At the scale of interest to us here, it is not possible to resolve each individual ice floe. We, thus, have to find an approximation which captures the smaller scale phenomena characteristic of individual floes interacting with each other. Although by no means the only option, the traditional fluids equations provide a natural starting point. (The other popular choice is to model the ice as a granular material.) We are here then deriving *continuous* equations to approximate a *discrete* phenomenon, but as the goal is a numerical model, we will ultimately *discretize* them again – just on a different scale.

Let us define the following variables:

$$
\begin{aligned}
h &= \text{average thickness of the ice} \\
c &= \text{concentration of ice (\% of sea surface area covered by ice)} \\
\mathbf{u} &= (u, v) = \text{horizontal velocities} \\
S &= \text{sources - sinks of ice (melting and freezing)} \\
F^x &= \text{sum of zonal forces (Coriolis, wind, currents, sea surface tilt, pressure)} \\
F^y &= \text{sum of meridional forces (Coriolis, wind, currents, sea surface tilt, pressure)} \\
[\rho &= \text{density of ice (a small correction, assumed in the following to be 1)}]
\end{aligned}
$$

In Eulerian coordinates, the mass conservation and momentum conservation equations can be written as

$$
\begin{aligned}
(ch)_t + \nabla \cdot (ch\mathbf{u}) &= S \\
(chu)_t + \nabla \cdot (chu\mathbf{u}) &= F^x \\
(chv)_t + \nabla \cdot (chv\mathbf{u}) &= F^y
\end{aligned}
$$

where the product ch essentially plays the role of a density in the analogy with the traditional fluids equations.

What differentiates the ice case from the standard fluids case is that in addition to these two conservation laws, we also have the constraint that c can be at most 1, which is enforced by a pressure force.

We will first consider the one-dimensional case without sources or sinks. By defining $F = F^x/(ch)$ and subtracting the first from the second equation, the equations reduce to:

$$\begin{aligned} (ch)_t + (chu)_x &= 0 \\ u_t + \left(\frac{u^2}{2}\right)_x &= F \end{aligned}$$

1.2 Lagrangian Formulation

To simplify the equations further, we introduce the Lagrangian coordinates

$$\begin{cases} \xi = \int_0^x ch\, d\hat{x} \\ \tau = t \end{cases}$$

Claim: The resulting Lagrangian equations in these coordinates for 1-D ice motion without sources or sinks are

$$\begin{cases} u_\xi = \left(\frac{1}{ch}\right)_\tau \\ u_\tau = F \end{cases}$$

Derivation:

- $\frac{\partial \xi}{\partial x} = ch$
  $\frac{\partial \xi}{\partial t} = \int_0^x (ch)_t \, d\hat{x} = -\int_0^x (chu)_{\hat{x}} d\,\hat{x} = -chu$

  $\frac{\partial \tau}{\partial x} = 0$
  $\frac{\partial \tau}{\partial t} = 1$

- $x = x(\xi(x,t), \tau(t))$
  $1 = \frac{\partial x}{\partial x} = x_\xi \xi_x + x_\tau \tau_x = x_\xi \xi_x$
  $\Rightarrow \frac{\partial x}{\partial \xi} = \frac{1}{\xi_x} = \frac{1}{ch}$

  $0 = \frac{\partial x}{\partial t} = x_\xi \xi_t + x_\tau \tau_t = \frac{\xi_t}{\xi_x} + x_\tau$
  $\Rightarrow \frac{\partial x}{\partial \tau} = -\frac{\xi_t}{\xi_x} = \frac{chu}{ch} = u$

  $t = t(\xi(x,t), \tau(t))$
  $0 = \frac{\partial t}{\partial x} = t_\xi \xi_x + t_\tau \tau_x = t_\xi \xi_x$
  $\Rightarrow \frac{\partial t}{\partial \xi} = 0$

  $1 = \frac{\partial t}{\partial t} = t_\xi \xi_t + t_\tau \tau_t = t_\tau$
  $\Rightarrow \frac{\partial t}{\partial \tau} = 1$

- $\frac{\partial u}{\partial \xi} = u_t t_\xi + u_x x_\xi = \frac{u_x}{ch}$
  $\left(\frac{1}{ch}\right)_\tau = -\frac{1}{(ch)^2}\left[(ch)_t t_\tau + (ch)_x x_\tau\right] = -\frac{1}{(ch)^2}\left[(ch)_t + (ch)_x u\right] = -\frac{1}{(ch)^2}\left[-chu_x\right] = \frac{u_x}{ch}$
  $\Rightarrow u_\xi = \left(\frac{1}{ch}\right)_\tau$

- $\frac{\partial u}{\partial \tau} = u_t t_\tau + u_x x_\tau = u_t + u_x u = F$
  $\Rightarrow u_\tau = F$

We bring these equations into a nicer form by introducing the variable $\rho \equiv \frac{1}{ch} - 1$, so that we end up with

$$\begin{cases} u_\tau &= F \\ \rho_\tau &= u_\xi \end{cases}$$

While these equations have a beguilingly simple form, we have now lost sight of the crucial variable c, which we have to constrain. We would like to translate this constraint into a constraint on ρ. This is easily done, provided $h \equiv 1$, and so we will concentrate on this particular case. The constraint $c \leq 1$ is now $\rho \geq 0$. Note that in addition to excluding melting and freezing processes, we have now also eliminated crushing from the problem. Obviously, to arrive at a realistic and versatile ice model, we will ultimately have to bring these components back. For now, however, we are concentrating on investigating the nature of the pressure force and its role in ice dynamics. For this purpose we make one more simplification and isolate the pressure term by setting all other forces equal to zero.

What form this pressure term should take is not immediately obvious. Various different suggestions have been made over the years. Our physical intuition tells us that the pressure does not act unless it is necessary to prevent ice concentration from exceeding 1, i.e. to prevent two ice floes from occupying the same space. In other words, while c is far from 1, $F_p = 0$ and the flow follows $u_\tau = 0$ (Each parcel's velocity does not change with time.). By adding a pressure to the system, we would like to deviate as little as possible from this route while satisfying the constraint. This suggests a mathematical formulation with the pressure defined as a Lagrange multiplier.

1.3 Pressure as Lagrange Multiplier

Note on Notation: *For the rest of this chapter, we will be discussing only the Lagrangian formulation. We will, thus, without loss, simplify our notation by replacing the Greek letters for the independent variables by their Roman cousins again, i.e. by writing the Lagrangian mass coordinate ξ as x and the Lagrangian time coordinate τ as t.*

The problem is to	minimize	u_t,
	given	$\rho_t = u_x$,
	subject to the constraint	$\rho \geq 0$.

Let us discretize this in time. We now need to

minimize	$\lvert u^{n+1} - u^n \rvert$ (or, equivalently, $(u^{n+1} - u^n)^2$) $\forall n \geq 1$,
given	$\rho^{n+2} = \rho^n + 2\Delta t\, u_x^{n+1} \;\; \forall n \geq 1$,
subject to the constraint	$\rho^m \geq 0 \;\; \forall m \geq 1$.

It is straight forward to see that the overall minimum will be achieved when each individual time step is minimized. We will look at a single time step.

Let $\begin{cases} v = u^{n+1} \\ u = u^n \\ \rho = \rho^n \end{cases}$

Our task is then to minimize $(v-u)^2$, subject to the constraint $\rho + 2\Delta t\, v_x \geq 0$.

The variational principle states for Lagrangian multiplier λ

$$\delta \int \left\{(v-u)^2 + \lambda(x)(\rho + 2\Delta t\, v_x)\right\}\, dx = 0$$

$$\begin{aligned}
\iff \forall\phi \quad 0 &= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \int \left\{(v+\epsilon\phi-u)^2 + \lambda\left(\rho + 2\Delta t\, \frac{\partial}{\partial x}(v+\epsilon\phi)\right)\right\} dx \\
&= \int \{2v\phi - 2u\phi + 2\Delta t\, \lambda\phi_x\}\, dx \\
&= \int 2\phi\{v - u - \Delta t\, \lambda_x\}\, dx \\
\therefore v - u &= \Delta t\, \lambda_x \quad \iff \quad \frac{v-u}{\Delta t} = \lambda_x
\end{aligned}$$

The equivalent continuous statement would be

$$u_t = \lambda_x$$

We will adopt the convention to write the pressure as $p = -\lambda$. Our final system for the Lagrangian formulation of 1-D sea ice dynamics without freezing and melting, crushing or external forces is thus

$$\begin{cases} u_t &= -p_x \\ \rho_t &= u_x \\ \rho &\geq 0 \end{cases}$$

1.4 Minimal Pressure

We have just derived a form for the pressure term as the x-derivative of a Lagrange multiplier. While this gives us the nice result that the pressure looks very much like the pressure for incompressible fluids, we still have little information about what the pressure actually is.

A constitutive law for frozen water can, theoretically, provide us with a way to calculate the pressure within an ice floe, as long as we know the external forces, including the pressure applied by other ice floes. Since we do not want to resolve the scale of individual ice floes, however, we have to find a different method.

Returning to the physical intuition that pressure only appears when it is called on to preserve $c \leq 1$ or, equivalently, $\rho \geq 0$, it seems reasonable that the pressure that appears in such a case is the minimum force that will do the job. Ice floes do not bounce (significantly) off each other. It is unphysical to expect the pressure two ice floes exert on each other to be greater than that which will halt further convergence.

It remains to answer in what norm we want p to be minimized. There are several reasonable choices. In particular, one might want to consider (a) the one-norm of p (in discrete form $\sum_{j=1}^{J} |p_j|$), (b) the two-norm of p ($\sum_{j=1}^{J} |p_j|^2$), (c) the one-norm of p_x ($\sum_{j=1}^{J} |p_{j+1} - p_j|$), or (d) the two-norm of p_x ($\sum_{j=1}^{J} |p_{j+1} - p_j|^2$). Analysis of a simple toy problem with known solution has shown that any of these four actually leads to the same answer. We will therefore choose the easiest of these to calculate, namely choice (a), the one-norm of p.

1.5 Numerics

The scheme is set up on a staggered grid, where ρ and p are defined at full steps, whereas u is defined at the half-steps in between.

In order to enforce the constraint that ρ stay non-negative, we have to advance the mass equation implicitly. As there is no evolution equation for p, it is irrelevant whether we advance u implictly or explicitly. Here we choose to write it implicitly, letting $p = 0$ originally. The discretizations are then

$$\begin{aligned} \rho_j^{n+1} &= \rho_j^n + \frac{\Delta t}{\Delta x}\left(u_{j+\frac{1}{2}}^{n+1} - u_{j-\frac{1}{2}}^{n+1}\right) \\ u_{j+\frac{1}{2}}^{n+1} &= u_{j+\frac{1}{2}}^{n} + \frac{\Delta t}{\Delta x}\left(p_j^{n+1} - p_{j+1}^{n+1}\right) \end{aligned}$$

Substituting from the second equation into the first, we get

$$\rho_j^{n+1} = \rho_j^n + \frac{\Delta t}{\Delta x}\left(u_{j+\frac{1}{2}}^{n} - u_{j-\frac{1}{2}}^{n}\right) - \left(\frac{\Delta t}{\Delta x}\right)^2 \left(p_{j+1}^{n+1} - 2p_j^{n+1} + p_{j-1}^{n+1}\right)$$

This equation is the constraint subject to which we want to minimize the one-norm of p. We also have two sets of constraint inequalities, namely $\rho \geq 0$ and $p \geq 0$. Since we have a linear system for p_j, we can use a simplex method to solve this optimization problem.

1.6 Postscript: The Program

The program comes in three parts

1. `ice_simplex.m`
   This is the main program. The input variables are dx = spatial step, $\mu = \frac{dt}{dx}$, and T = max time. The output variables are u, p, ρ, x (the spatial grid for ρ and p), $x1$ (the spatial grid for u), and t (temporal grid). The first three outputs are matrices.

   A call to this program from Matlab might look like this:

   ```
   [u,p,rho,x,x1,t]=ice_simplex(.01,.5,1);
   ```

   The domain is periodic on [0,1]. Initial conditions are $\rho = 0.5$ and $u = \sin(2\pi\, x)$.

   Along the way, this program calls on simplex.m.

2. `simplex.m`
   This program performs the optimization step. It minimizes $z = cv + d$, subject to the constraints $Av = b$ and $v \geq 0$. It takes as input c, d, A and b, and returns as output the minimizing v and the minimal z.

   When called from `ice_simplex.m`, v consists of p_j and ρ_j, while z gives the norm of p.

   Along the way, this program calls on pivot.m.

3. `pivot.m`
   This is a technical program performing the pivoting task called on repeatedly by simplex.m. That is its raison d'être.