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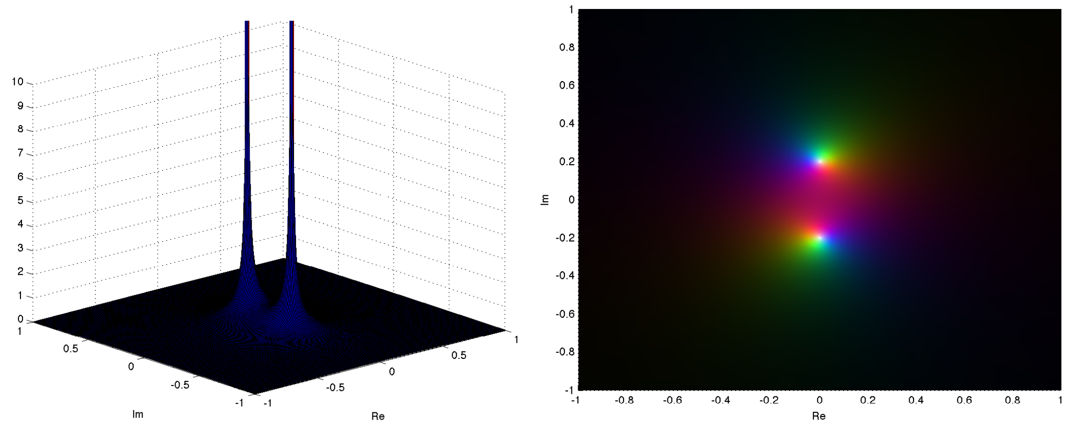
1/30/12
Math 126 HW 3

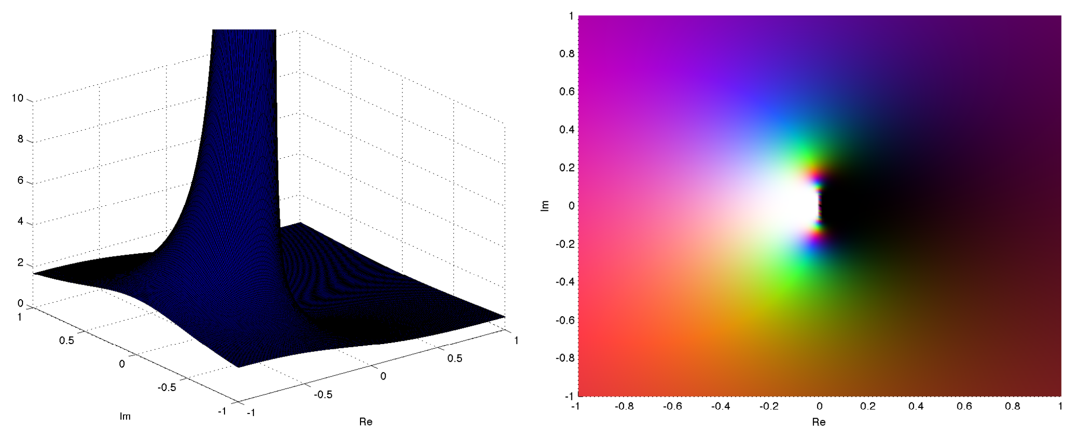
1. Code: complex_vis.m

   (a) $f(z) = e^{-1/z}$ singularities are at $z = \pm\frac{1}{5}i$. Both poles are simple.



left: Plot of $f(z) = (1 + 25z^2)^{-1}$, where z-axis is $|f(z)|$, right: black is vanishing, white is $|f(z)| = \infty$, colors correspond to contribution of real and imaginary components to $|f(z)|$. Red is positive real, green is negative imaginary, blue is positive imaginary.

   (b) $f(z) = e^{-1/z}$ singularity at $z = 0$. Pole is of infinite order, as expanding $f(z)$ as $e^{-1/z} = 1 + (\frac{-1}{z}) + \frac{1}{2!}(\frac{-1}{z})^2 + ...$ has an infinite number of terms with $(\frac{1}{z})^n, n = 0, 1, 2, ...$



left: Plot of $f(z) = e^{-1/z}$, where z-axis is $|f(z)|$, right: black is vanishing, white is $|f(z)| = \infty$, colors correspond to contribution of real and imaginary components to $|f(z)|$. Red is positive real, green is negative imaginary, blue is positive imaginary.

2. Prove that, given a set of distinct points $\{x_j\}_{j=0,...n}$ in $[a, b]$ there exists a *unique* set of weights $\{w_j\}_{j=0,...,n}$ such that Newton-Cotes quadrature integrates exactly over $[a, b]$ all polynomials up to degree n.

   Consider a degree-n polynomial $p(x) = a_0 + a_1x + ... + a_nx^n$. We can consider this polynomial as a linear combination of the monomials $1, x, x^2, ..., x^n$, which are all linearly independent.

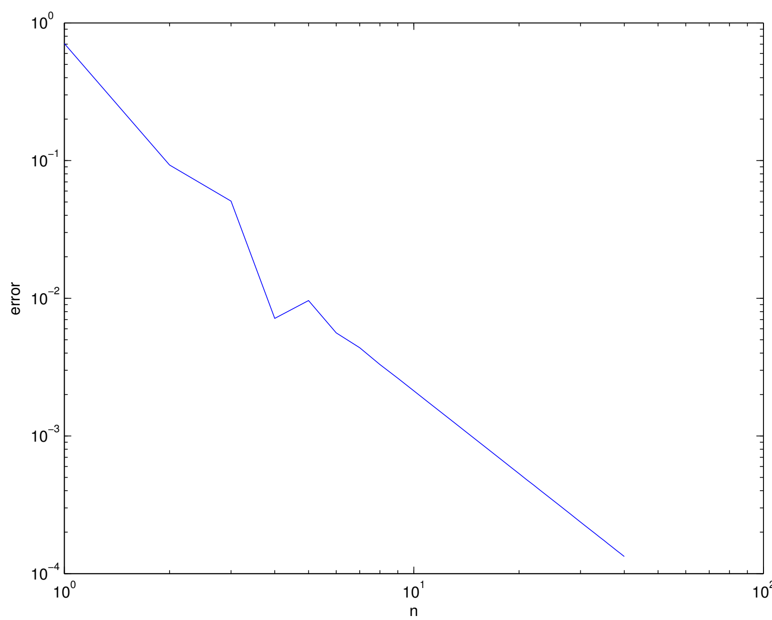
To integrate each monomial exactly over $[a, b]$ using Newton-Cotes quadrature, we have

$$\sum_{i=0}^{n} w_i f(x_i) = \int_a^b f(x)dx$$
$$\sum_{i=0}^{n} w_i a_k x_i^k = \frac{1}{k+1} a_k (b^{k+1} - a^{k+1})$$
$$\sum_{i=0}^{n} w_i x_i^k = \frac{1}{k+1} (b^{k+1} - a^{k+1})$$

Since there are $n+1$ distinct x_i, we can create a $k+1 \times n+1$ matrix to solve for the exact weights w_i which will work for all polynomials up to degree $k-1$ (since the a_k cancelled out above). We can go up to $k = n$ for an $n+1 \times n+1$ matrix which will be invertible, as the x_i are unique leading the matrix to be full rank. Thus, we can solve the system $\mathbf{A}\mathbf{w} = \mathbf{b}$ exactly, where $\mathbf{A}$ has x_i^k in the kth row and ith column, $\mathbf{w}$ is the column vector of w_is, and $\mathbf{b}$ is the column vector where the ith entry is the integral of the ith monomial over $[a, b]$. Thus there exists a unique set of weights given a set of distinct points $\{x_j\}_{j=0,\ldots n}$in $[a, b]$ that will integrate all polynomials up to degree n exactly. If we try to find a set of weights for degree greater than n, the matrix will no longer be full-rank, and we will not be able to integrate exactly.

3. Numerical integration of $(1 + 4x^2)^{-1}$ on [-1,1] (exact answer is arctan(2)). Code: quad.m

   (a) Trapezoid rule

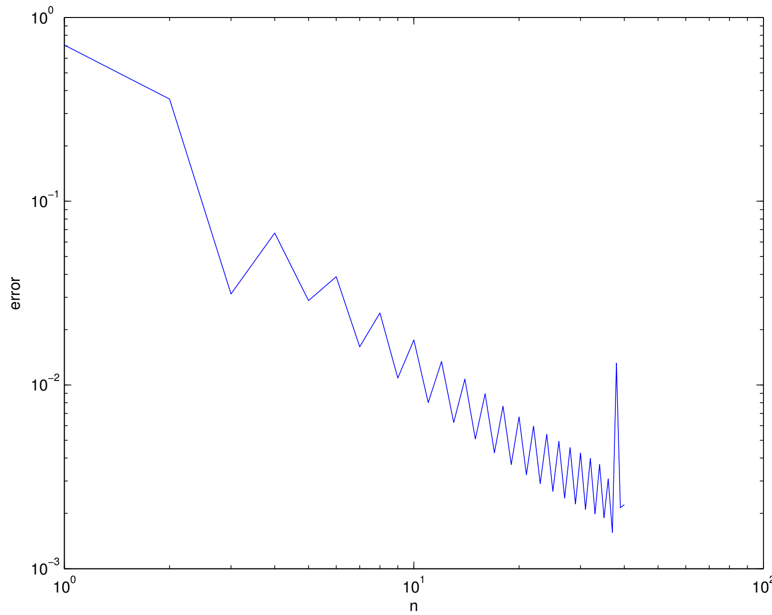


Error of numerical integration by trapezoid rule as function of n+1 points on a log-log graph.

Using Thm. 9.4 from Kress, we know that the error using the trapezoid rule is bounded by $\frac{2}{3}||f''(x)||_\infty$ Thus for an arbitrary n, we have the error bounded by $n\frac{2/n^3}{3}||f''(x)||_\infty = \frac{2}{3n^2}||f''(x)||_\infty$. $||f''(x)||_\infty = 8$ (using wolframalpha), so the error is bounded by $16/3n^2$.

Using the graph, we find that the error is proportional to $\approx n^{-8/3}$, which is within the bounds of the theorem

(b) Newton-Cotes

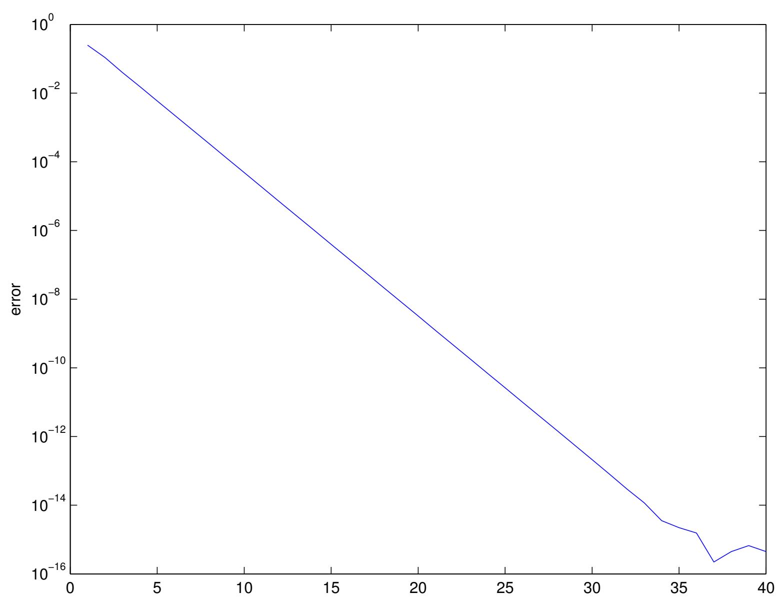


Error of numerical integration by Newton-Cotes method as function of n+1 points on a log-log graph.

Minimum achievable error is just above 10^{-3}. As seen on the graph, error starts to blow up just before $n = 40$ because the Vandermonde matrix used to solve for weights becomes badly scaled.

4. Gaussian Quadrature on $[-1, 1]$ Code: gaussquad.m

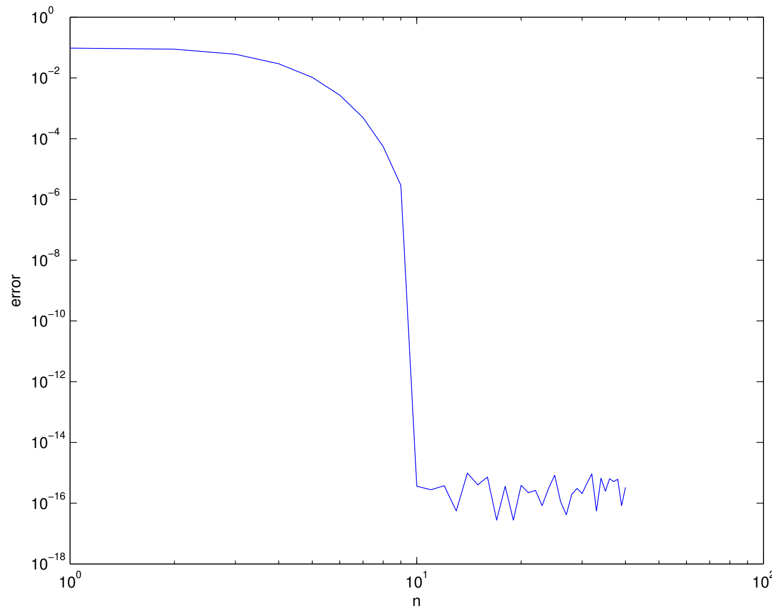
(a) $(1 + 4x^2)^{-1}$ (answer: arctan(2))



Error of numerical integration by gaussian quadrature as function of n+1 points on a semilog graph.

Convergence is exponential. For $E = Ce^{-\alpha n}, \alpha \approx 1.04$. Since function is analytic within a region about $[-1, 1]$ on the real line, we get exponential convergence, which is good.

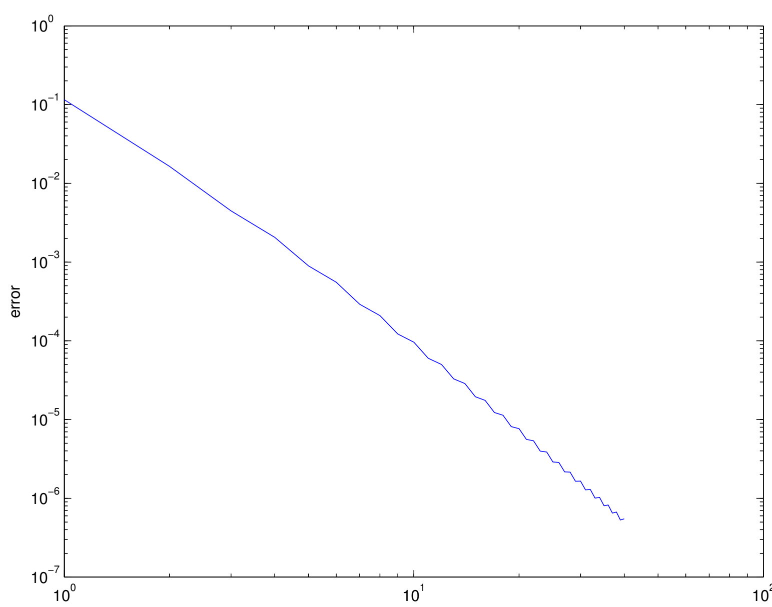
(b) x^{20} (answer: 2/21)



Error of numerical integration by gaussian quadrature as function of n+1 points on a log-log graph.

Convergence looks super-algebraic. By Kress Thm. 9.20, the convergence is proportional to $f^{(2n+2)}$ for some point on the interval which is just 0 for $n \geq 10$, as seen on the graph (the error goes to 10^{-16} because of machine imprecision).

(c) $|x|^3$ (answer: 1/4)



Error of numerical integration by gaussian quadrature as function of n+1 points on a log-log graph.

Convergence is algebraic, of order ≈ -3.33. Although the function is not continuous differentiable, it is fairly well behaved (no singularities), so we get algebraic convergence (instead of exponential).

5. Base case: Using $q_{-1} = 0, q_0 = 1$, use $q_{j+1}(x) = xq_j(x) - \alpha_{j+1}q_j(x) - \beta_{j+1}q_{j-1}(x)$ where $\alpha_{j+1} := (q_j, xq_j)/(q_j, q_j)$ and $\beta_{j+1} := (q_j, q_j)/(q_{j-1}, q_{j-1})$, except $\beta_1 = 0$. Constructing q_1, we have

$$\begin{aligned} q_1 &= x - \frac{\int_{-1}^{1} x dx}{\int_{-1}^{1} 1 dx} - 0 \\ &= x - 0 \\ &= x \end{aligned}$$

Taking the inner product $(q_1, q_0) = \int_{-1}^{1} x dx = 0$, we see that the set $\{q_0, q_1\}$ is mutually orthogonal.

Inductive Case: Let $\{q_0, ..., q_j\}$ be a mutually orthogonal set of polynomials on $[-1, 1]$. Then suppose $q_{j+1}(x) = xq_j(x) - \alpha_{j+1}q_j(x) - \beta_{j+1}q_{j-1}(x)$ where $\alpha_{j+1} := (q_j, xq_j)/(q_j, q_j)$ and $\beta_{j+1} := (q_j, q_j)/(q_{j-1}, q_{j-1})$. Then taking the inner product of q_i with q_{j+1}, $i \leq j$, we have,

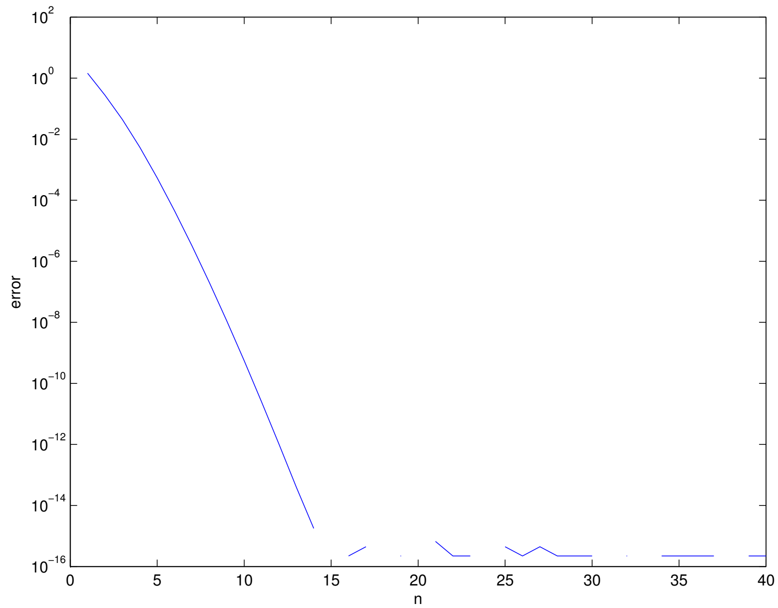
$$(i, j+1) = (i, xj) - \frac{(j, xj)}{(j, j)}(i, j) - \frac{(j, j)}{(j-1, j-1)}(i, j-1)$$

Clearly, when $i = j$, the third term goes to 0 (by orthogonality), and the first and second terms cancel out, leaving $(j, j+1) = 0$. If $i = j - 1$, then we take the first term as $(j-1, xj) = (j, x(j-1)) = (j, j + \alpha_j(j-1) + \beta_j(j-2)) = (j, j)$ using orthogonality of the set. Thus, $(j-1, j+1) = (j, j) - \frac{(j,j)}{(j-1,j-1)}(j-1, j-1) = (j, j) - (j, j) = 0$. For $i \neq j, j-1$, we can modify the previous relation for $(j-1, xj)$ to have $(i, xj) = (j, (i+1) + \alpha_i i + \beta_i(i-1)) = 0$ since $i < j - 1$. Thus $(i, j+1) = 0$, and the term q_{j+1} is orthogonal to the set $\{q_0, ..., q_j\}$.

By induction we see that the given rules construct a sequence of orthogonal polynomials on $[-1, 1]$.

6. Error convergence over $[0, 2\pi)$ using periodic trapezoid rule. Code: periodquad.m

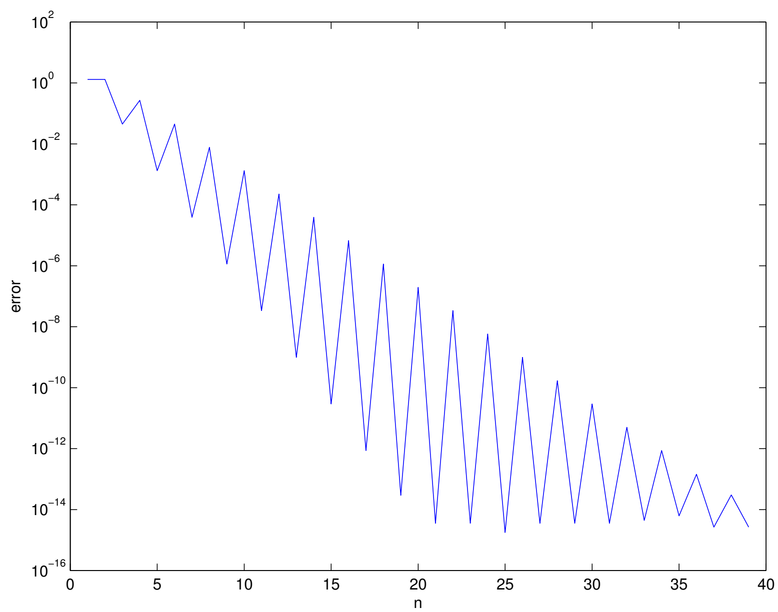
   (a) $(1/2\pi)e^{\cos x}$ exact answer: modified Bessel function $I_0(1)$



Error of numerical integration by periodic trapezoid rule as function of n+1 points on a semilog graph.

Function is C^∞ so we expect super-algebraic convergence. It is kind of difficult to see if this is happening on the plot, as the convergence is so fast (goes to $O(\epsilon_{mach})$ for $n = 15$

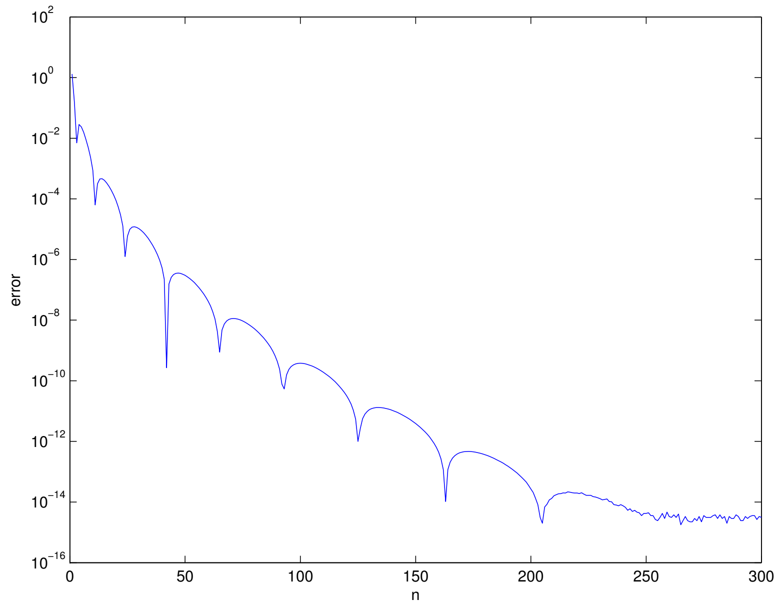
(b) $(1 + \cos^2(x/2))^{-1}$ numerical answer: 4.44288293815837



Error of numerical integration by periodic trapezoid rule as function of n+1 points on a semilog graph.

Convergence is exponential, for $E = Ce^{-\alpha n}, \alpha \approx 0.933$. Singularities for the function are at $z = 2(2\pi n \pm \cos^{-1}(i))$, Since the function is analytic within a domain about the real line, the series converges exponentially.

(c) $\exp(-1/|\sin(x/2)|)$ numerical answer: 1.31314591268447



Error of numerical integration by periodic trapezoid rule as function of n+1 points on a semilog graph.

Function is not analytic, nor continuously differentiable, so we don't see exponential or algebraic convergence.