Brad Nelson
2/6/12
Math 126 HW 4
Collaborator: Vipul Kakkad

1. (a) (follows NA lemma 9.16) From NA lemma 9.14, we know that if $\{x_0, \ldots, x_n\}$ are gaussian nodes on $[a, b]$, then $q_{n+1} = \prod_{i=0}^{n}(x - x_i)$ is orthogonal to all $p \in \mathbb{P}_n$ on $[a, b]$. Now, suppose that not all nodes are simple. Then let $\{y_1, \ldots, y_m\}$ be the set of all points for which q_n changes sign. Since not all nodes are simple, $m < n$. We can construct a polynomial $r_m = \prod_{i=1}^{m}(x - y_i)$ which is of order $m < n$, thus, $r_m \in \mathbb{P}_{n-1}$ and $q_n \perp r_m$. Because of orthogonality, we expect $\int_a^b p_n r_m dx = 0$, but because of the way we have chosen r_m, the product never changes sign. Since neither function is 0, we know that the integral is non-zero. This leads to a contradiction. Thus, the polynomial p_n that defines the gaussian nodes is non-simple.

   (b)

$$\begin{aligned}
||K||_2^2 &= \max_{||u||_2=1} \int_a^b dt \Big(\int_a^b ds k(t,s) u(s) \Big)^2 \\
&\leq \max_{||u||_2=1} \int_a^b dt ||k(t)||_{2_s}^2 \cdot ||u||_2^2 && \text{Cauchy-Schwarz} \\
&\leq \int_a^b dt ||k(t)||_{2_s}^2 && \text{since } ||u||_2 = 1 \\
&\leq ||k||_2^2 && \text{2-norm on } [a,b]^2
\end{aligned}$$

2. Solve analytically:

$$u(t) + \int_0^1 ts^3 u(s) ds \;=\; 1, \qquad \text{for } t \in [0,1]$$

Since the range of the integral operator is t, we know that $u(t)$ will be of the form $1 + Ct$, $C \in \mathbb{R}$.

$$\begin{gathered}
(1 + Ct) + t \int_0^1 s^3 (1 + Cs) ds = 1 \\
Ct + t \Big[\tfrac{1}{4} s^4 + \tfrac{C}{5} s^5 \Big]_0^1 = 0 \\
C = \tfrac{-1}{4} - \tfrac{C}{5} \\
C = \tfrac{-5}{24}
\end{gathered}$$

Thus,

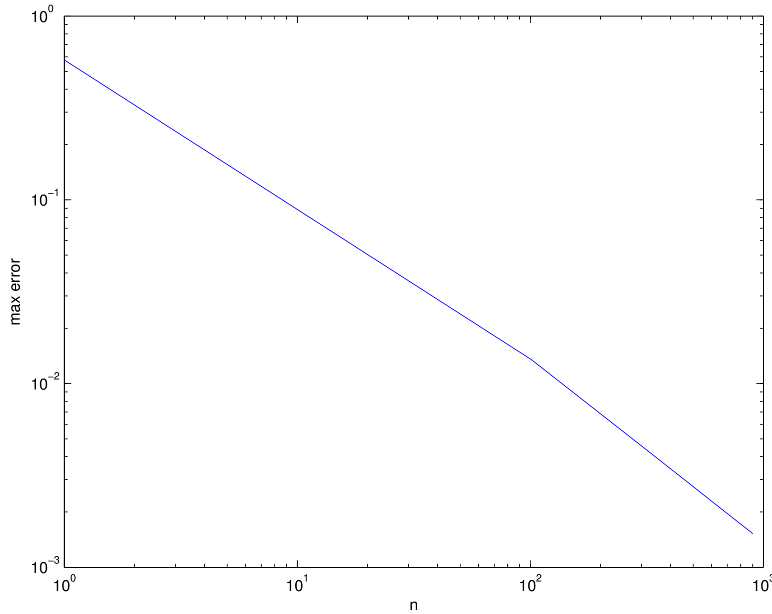
$$u(t) = 1 - \frac{5}{24} t$$

The operator is compact since it has a continuous kernel (NA Thm 12.4)
By NA Thm 12.5, the infinity-norm for this operator is given by

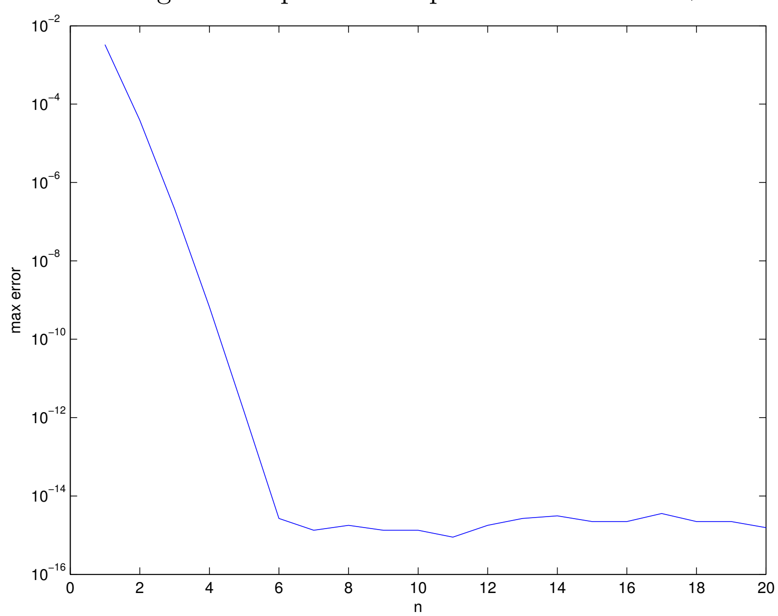
$$||K||_\infty = \max_{0 \leq t \leq 1} \int_0^1 |k(t,s)| ds$$

The integral evaluates to $t/4$ which has a max of $1/4$ at $t = 1$. Thus $||K||_\infty = 1/4$.

3. (a) code: nystrom.m



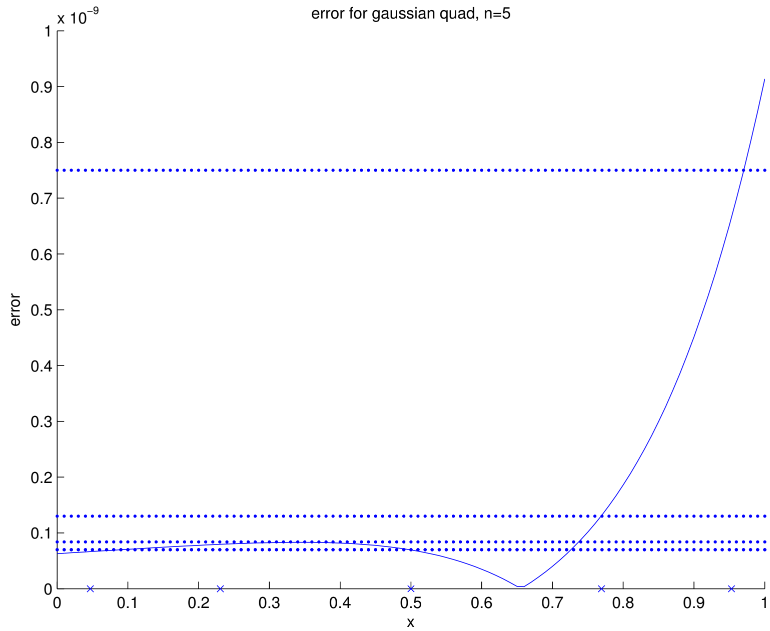
above: convergence for periodic trapezoid scheme for n+1 nodes



above: convergence for gaussian scheme for n+1 nodes

For error$< 10^{-5}$, n=3 for gausssian quadrature, and $n \approx 10^5$ for periodic trapezoid quadrature (extrapolating from plot). Convergence is exponential for gaussian quadrature (straight on a semilogy plot). This is the same as convergence for the quadrature scheme (see hw.3). Convergence is algebraic for periodic trapezoid quadrature (straight on a loglog plot). This is same as convergence for the quadrature scheme (see hw.3).

(b) For the system $Ax = b$, in this problem $A = (I - K)$ is an $N \times N$ matrix. The kernel, $k(t, s) = e^{ts}$ is between 1 and e on $[0, 1]^2$, so the entries of the K matrix are bounded between 1 and e multiplied by the weights of the functions, which go to 0 as n increases. So, as N increases, $I - K$ approaches I. Thus, the condition number of the matrix tends to 1.

(c) 

Error of interpolated solution for N=5 for Gauss nodes. Error at nodes in horizontal lines.

Sup norm of error is greater than max error at gauss nodes (at far right). Not as assumed in part (a).

4. (a) let $Ku = \int_0^{2\pi} (1/2\pi) k(t-s) u(s) ds$ be a periodic convolution operator with a 2π periodic kernel. Let $u = e^{imt}, m \in \mathbb{Z}$.

$$\begin{aligned} Ku(t) &= \frac{1}{2\pi}\int_0^{2\pi} k(t-s) e^{ims} ds \\ &= \frac{1}{2\pi}\int_{0-t}^{2\pi-t} k(x) e^{im(t-x)} dx && x = t-s \\ &= e^{imt}\frac{1}{2\pi}\int_0^{2\pi} k(x) e^{-imx} && \text{since functions are } 2\pi \text{ periodic, no } t \text{ in bounds} \end{aligned}$$

Thus, e^{imt} is an eigenfunction of K, with eigenvalue $\frac{1}{2\pi}\int_0^{2\pi} k(x)e^{-imx}$

(b) Consider $Ku = f$. Now, write k, u, and f as Fourier series.

$$\frac{1}{2\pi}\int_0^{2\pi} \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} k_m e^{im(t-s)} u_n e^{ins} ds = \sum_{k\in\mathbb{Z}} f_k e^{ikt}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} k_m e^{imt} e^{-ims} u_n e^{ins} ds = \sum_{k\in\mathbb{Z}} f_k e^{ikt}$$

$$\sum_{n\in\mathbb{Z}} k_n u_n e^{int} = \sum_{k\in\mathbb{Z}} f_k e^{ikt} \quad \text{by orthonormality of} \frac{1}{2\pi}\int_0^{2\pi} e^{-int} e^{imt} dt$$

Thus, we see that $k_n u_n = f_n \ \forall n \in \mathbb{Z}$

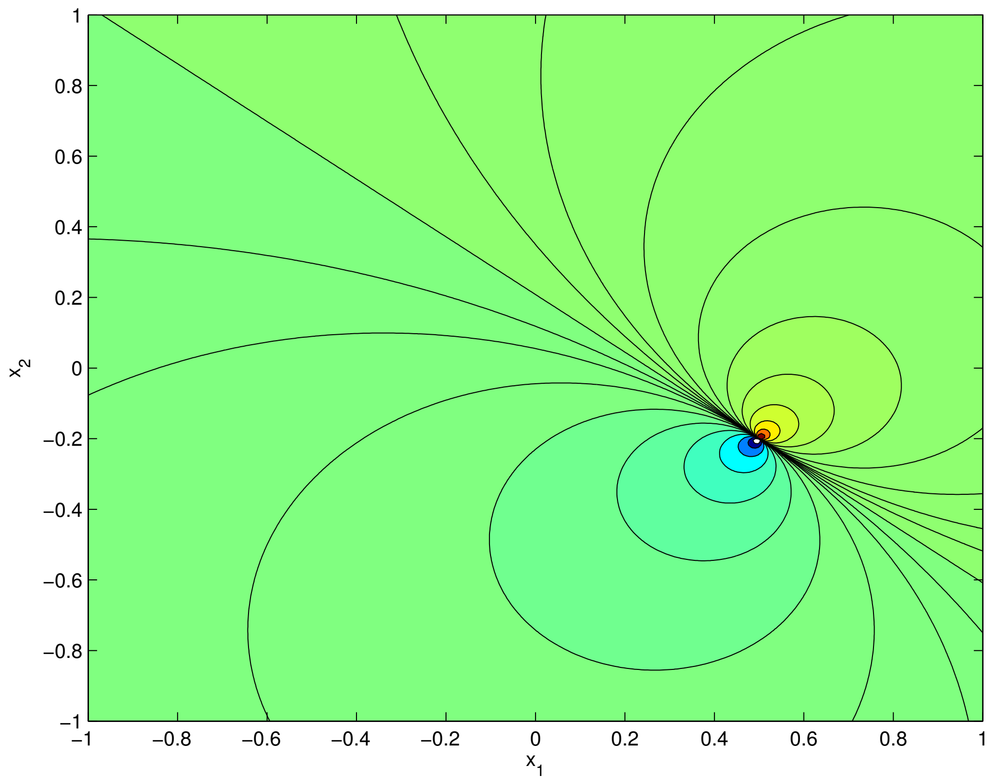
(c) $||K||_2 = \max_{||u||_2=1} ||Ku||$. Since the Fourier basis is complete, $\sum u_n^2 = 1$, and the largest that any u_n can be is 1. Using the result from part (b) that in the Fourier basis, $Ku = \sum_{n\in\mathbb{Z}} k_n u_n e^{int}$, and $||e^{int}||_2 = \sqrt{2\pi}$, we see that $||K||_2 = \max \sqrt{2\pi}|k_n|$.

(d) The condition number of $Ku = f$ goes to infinity since K^{-1} sets $u_n = f_n/k_n$, and $k_n \to 0$ as $n \to \infty$. The condition number of $u - Ku = f$ is $\max\{|\frac{1}{1-k_n}|, |1-k_n|\}$

5. Fundamental solution to Laplace Equation: $\Phi(x, y) = (1/2\pi)ln(1/|x-y|)$

(a) see laplacefs.m for code

(b) see lapfsplot.m for code



contour plot for $\partial\Phi/\partial n_y$, $y = (0.5, -0.2)$, $n_y = (1/2, \sqrt{3}/2)$, $x \in [-1, 1]^2$. contour intervals are 0 (diagonal), and intervals between $\pm 10^{-1.5}$ to $\pm 10^2$ logarithmically spaced. Values below diagonal are positive ($\to$ red), above diagonal are negative ($\to$ blue). Note singularity at x=y.