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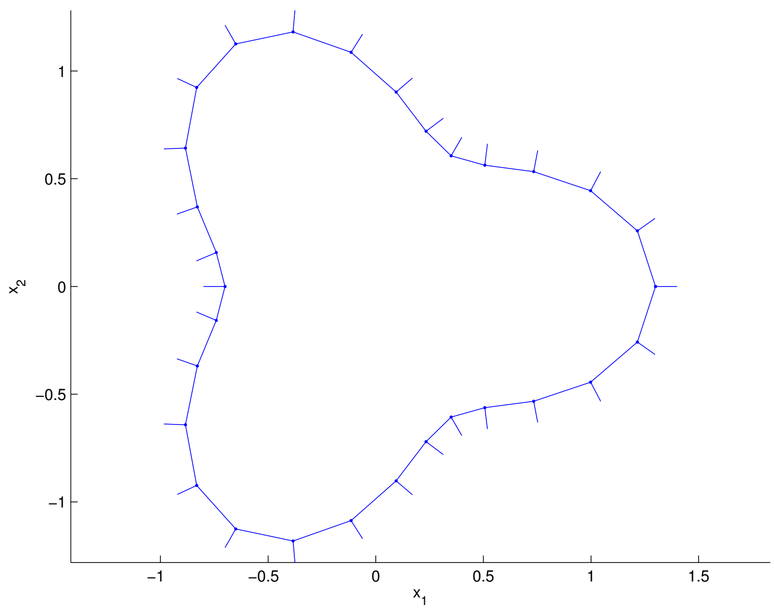
2/12/12
Math 126 HW 5

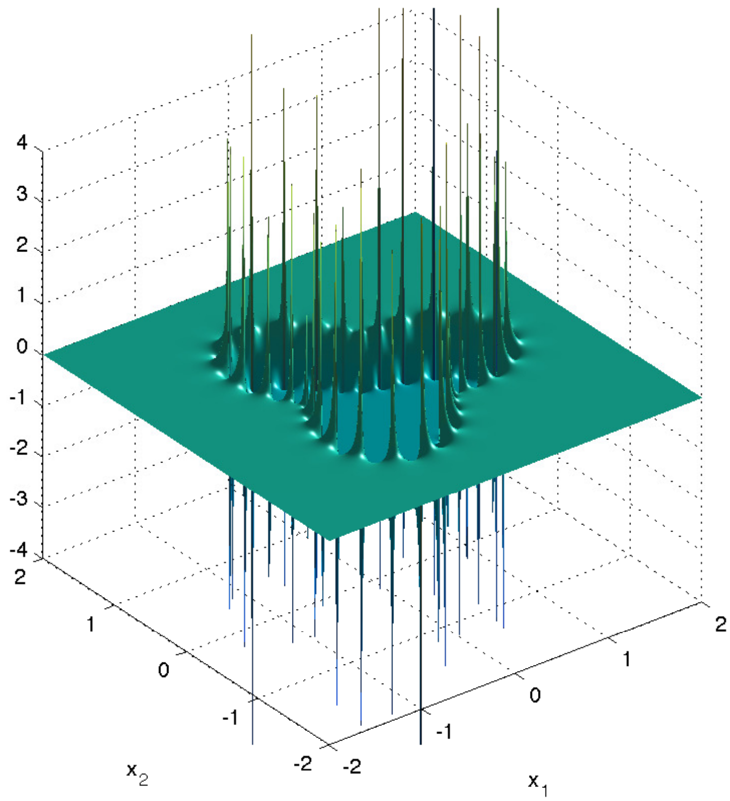
1. (a) Code inside: param.m
   (b) Code: param.m
   (c) Code: boundplot.m
   Also used: laplacefs.m




$R(s) = 1 + 0.3\cos(3s)$, $M = 30$ nodes. Normal vectors shown as lines tangent to curve at nodes.

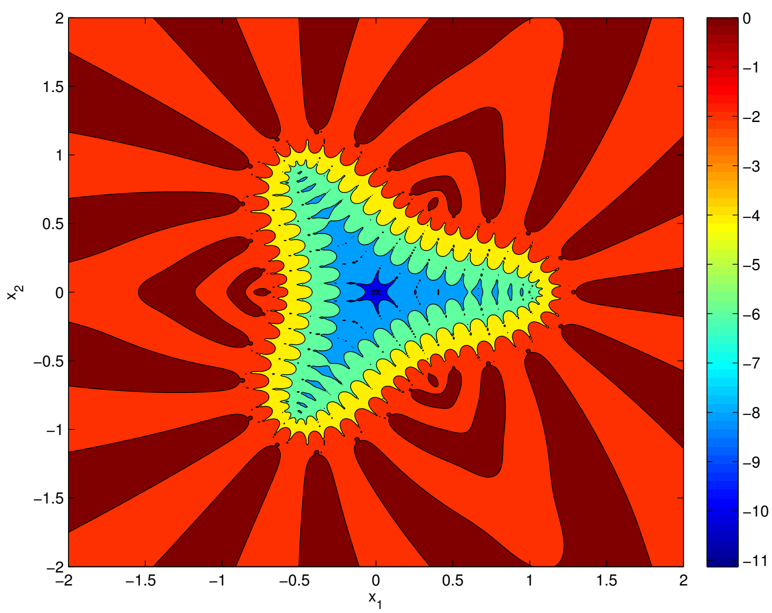
2. Code: gausslaw.m

   (a) $u = D\tau$, $\tau = -1$, periodic trapezoidal quadrature with 30 nodes on boundary.



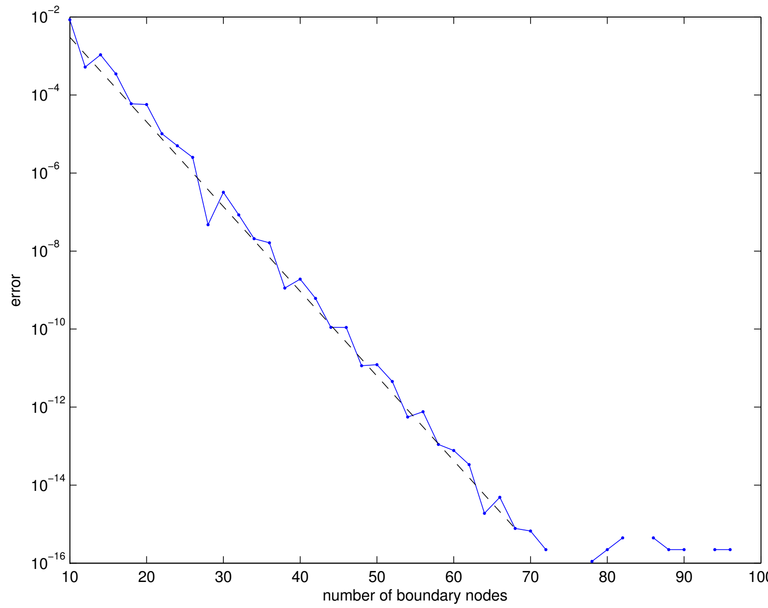

note how $u \approx 0$ outside boundary and $u \approx \tau$ inside boundary.

(b) Plot of $\log_{10}|u+1|$. Shows absolute error inside the domain, where $u \approx -1$, and $\approx 1$ outside boundary, where $u \approx 0$.




Error seems to exponentially decrease towards the interior of the domain.

(c) Error at a $x = (0.2, 0.1)$ vs. the number of boundary nodes used.




Note semilogy plot. Dashed line is $y = 0.45e^{-0.5n}$

Convergence is exponential (rate is $e^{-0.5n}$). Error bottoms out at about 75 nodes.

3. Proof of bound on the "far" part in the double-layer jump relation. Fix $y, z \in \partial\Omega$, and let $x = x(h) = z + hn_z$ be a point off the surface for $h \neq 0$. We make a geometric assumption $2h \leq |z - y|$.

   (a) Let $r_z = y - z, r_x = y - x$. Since $2h \leq |z - y|$, we have $\frac{1}{2}r_z \leq r_x \leq \frac{3}{2}r_z$, where the extreme cases occur when $n_z$ is parallel or antiparallel to $r_z$. We know that

$$\frac{\partial\Phi(x,y)}{\partial n_y} = -\frac{1}{2\pi}\frac{n_y \cdot (y-x)}{|y-x|^2}$$

Thus,

$$\begin{aligned}\left|\frac{d}{dh}\left(\frac{\partial\Phi(x,y)}{\partial n_y} - \frac{\partial\Phi(z,y)}{\partial n_y}\right)\right| &= \left|\frac{1}{2\pi}\frac{d}{dh}\left(\frac{n_y \cdot r_z}{r_z^2} - \frac{n_y \cdot r_x}{r_x^2}\right)\right| \\ &\leq \frac{1}{2\pi}\frac{d}{dh}\left(\left|\frac{n_y \cdot r_z}{r_z^2}\right| + \left|\frac{n_y \cdot r_x}{r_x^2}\right|\right) \\ &\leq \frac{1}{2\pi}\frac{d}{dh}\left(\left|\frac{n_y \cdot r_z}{r_z^2}\right| + \left|4\frac{n_y \cdot r_x}{r_z^2}\right|\right)\end{aligned}$$

Since $4\frac{1}{r_z^2} \leq \frac{1}{r_x^2} \leq \frac{4}{9r_z^2}$, and we are assuming that $r_x$ is similar in size to $r_z$, so the larger coefficient will give the extreme bound. Now, $n_y \cdot r_x = n_y \cdot (r_z + hn_z)$, which is largest in magnitude relative to $n_y \cdot r_z$ when $n_y \cdot n_z = 1$, so $n_y \cdot r_x \le n_y \cdot r_z + h$. Thus,

$$\begin{aligned}\left|\frac{d}{dh}\left(\frac{\partial\Phi(x,y)}{\partial n_y} - \frac{\partial\Phi(z,y)}{\partial n_y}\right)\right| &\le \frac{1}{2\pi}\frac{d}{dh}\left(\frac{|n_y \cdot r_z|}{r_z^2} + 4\frac{|n_y \cdot r_z| + h}{r_z^2}\right)\\ &\le \frac{1}{2\pi}\frac{d}{dh}\left(\frac{5|n_y \cdot r_z| + 4h}{r_z^2}\right)\\ &\le \frac{1}{2\pi}\left(\frac{4}{r_z^2}\right)\\ &\le \frac{2}{\pi r_z^2}\end{aligned}$$

Which is just $C/|z-y|^2$, where $C = 2/\pi$.

(b) Since $z$ is fixed, let

$$g(h,y) = \left|\frac{\partial\Phi(x,y)}{\partial n_y} - \frac{\partial\Phi(z,y)}{\partial n_y}\right|$$

and $S = y \in \partial\Omega, |y-z| \ge r$. Since $|y-z| \ge r$, and $h < r/2$, the relation that we found for part (a) holds, and we can even strengthen it since $1/r_z \le 1/r$, by stating $\frac{d}{dh}g(h,y) \le C/r^2$, $C = 2/\pi$. The integral becomes

$$\int_S g(h,y)dy \le g(h,y_f) - g(h,y_0) + \frac{1}{2}h\ell(S)\frac{d}{dh}g$$

Where $\ell(S)$ is the length of $S$, and $y_0$ and $y_f$ are the respective start and end points for $S$. Since $S$ is almost closed, $g(h,y_f) \approx g(h,y_0)$, so we have

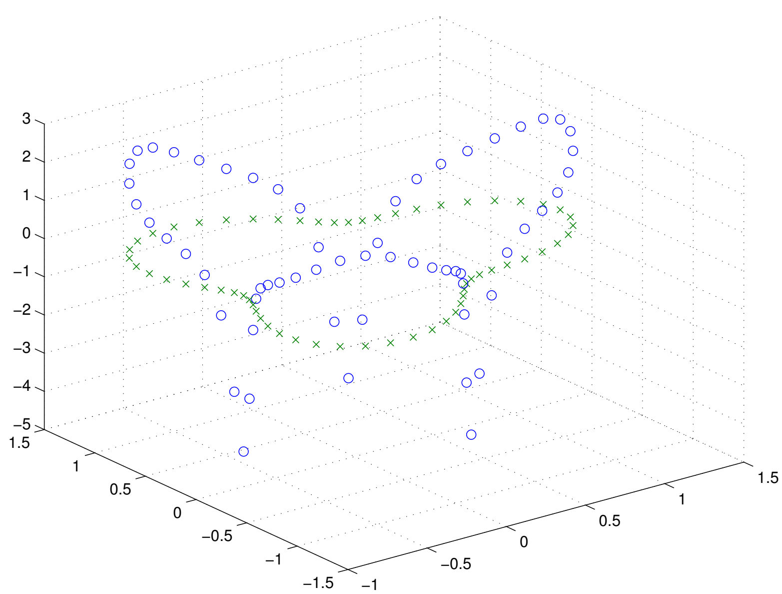
$$\int_S g(h,y)dy \le C\frac{h}{r^2}$$

where $C = \ell(S)/\pi$

4. Now to solve BVP $(D - \frac{1}{2})\tau = f$, or $A\tau = -2f$, where $A = I - 2D$.

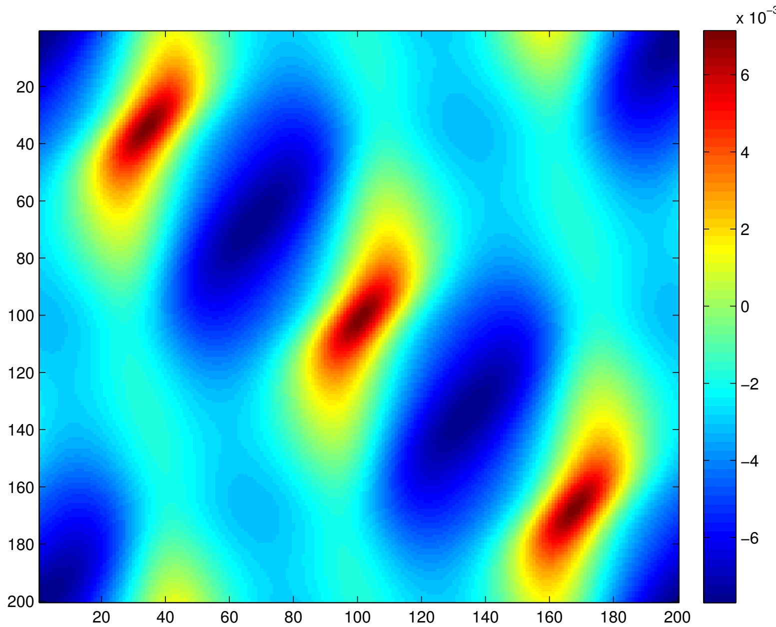
$$D\tau = \int_{\partial\Omega} \frac{\partial\Phi(x,y)}{\partial n_y}\tau(y)ds_y$$

(a) Code inside: param.m



Curvature at 60 nodes (blue), 2D surface in green.

(b) Code: bvp.m




Kernel for $D$ using 200 nodes. Note smooth transition over diagonal.

(c) Now we can use $V = D\tau$ inside $\Omega$ to find the double layer potential $V$ for some point $x \in \Omega$.
Code: bvp.m
We find that for $x = (0.2, 0.1)$, $u^{(n)}(x) = 1.083140928009776$ using 30 boundary nodes.
The error from the known solution, $u = \cos(x_1)e^{x_2}$, is $1.516 \times 10^{-7}$.