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BOUNDARY INTEGRAL EQUATIONS IN TIME-HARMONIC ACOUSTIC SCATTERING

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Abstract — We first review the basic existence results for exterior boundary value problems for the Helmholtz equation via boundary integral equations. Then we describe the numerical solution of these integral equations in two dimensions for a smooth boundary curve using trigonometric polynomials on an equidistant mesh. We provide a comparison of the Nyström method, the collocation method and the Galerkin method. In each case we take proper care of the logarithmic singularity of the kernel of the integral equation by choosing appropriate quadrature rules. In the case of analytic data the convergence order is exponential. The Nyström method is the most efficient since it requires the least computational effort. Finally, we consider boundary curves with corners. Here, we use a graded mesh based on the idea of transforming the nonsmooth case to a smooth periodic case via an appropriate substitution. Then, the application of Nyström's method again yields rapid convergence.

1. BOUNDARY INTEGRAL EQUATIONS

We begin with a brief review on the theory of boundary integral equations in time-harmonic acoustic scattering. For a comprehensive study we refer to [1]. Consider acoustic wave propagation in a homogeneous isotropic medium in $\mathbb{R}^2$ or $\mathbb{R}^3$ with speed of sound c. The wave motion can be determined from a velocity potential $U = U(x,t)$ from which the velocity field w is obtained by

$$w = \operatorname{grad} U,$$

and the pressure p by

$$p - p_0 = -\frac{\partial U}{\partial t},$$

where p_0 denotes the pressure of the undisturbed medium. In the linearized theory, the velocity potential U satisfies the wave equation

$$\frac{\partial^2 U}{\partial t^2} - c^2 \Delta U = 0.$$

Hence, for time-harmonic acoustic waves of the form $U(x,t) = u(x)\, e^{-i\omega t}$ with frequency ω, we deduce that the space dependent part u satisfies the *reduced wave equation* or *Helmholtz equation*

$$\Delta u + \kappa^2 u = 0,$$

where the wave number is given by $\kappa = \omega/c$. The mathematical description of the scattering of time-harmonic waves by an obstacle leads to exterior boundary value problems for the Helmholtz equation. For example, prescribing the values of u on the boundary of the obstacle physically corresponds to prescribing the pressure of the acoustic wave on the boundary. Therefore, scattering from sound-soft obstacles leads to a Dirichlet boundary value problem. Similarly, prescribing the normal derivative on the boundary corresponds to prescribing the normal velocity of the acoustic wave. Thus, scattering from sound-hard obstacles leads to a Neumann boundary value problem. For the sake of brevity we confine our presentation to the Dirichlet boundary condition.

Let $D_- \subset \mathbb{R}^m$, $m = 2, 3$, be a bounded domain with a connected boundary $\Gamma = \partial D_-$. By n we denote the unit normal to Γ directed into the exterior domain $D_+ := \mathbb{R}^m \setminus \overline{D}_-$. Then, the *exterior Dirichlet problem* consists in finding a solution u to the Helmholtz equation

$$\Delta u + \kappa^2 u = 0 \qquad \text{in } D_+ \tag{1.1}$$

satisfying the boundary condition

$$u = f \qquad \text{on } \Gamma \tag{1.2}$$

and the *Sommerfeld radiation condition*

$$\frac{\partial u}{\partial r} - i\kappa u = o(r^{(1-m)/2}), \qquad r = |x| \to \infty, \tag{1.3}$$

uniformly in all directions $x/|x|$. This radiation condition ensures the uniqueness for the boundary value problem, by Rellich's theorem.

For smooth boundaries, for example Γ of class C^2, i.e., Γ is twice continuously differentiable, the existence of a solution u of (1.1)–(1.3) can be based on boundary integral equations. In the so-called *layer approach*, we seek the solution in the form of acoustic surface potentials. The fundamental solution to the Helmholtz equation is given by

$$\Phi(x,y) := \begin{cases} \dfrac{i}{4} H_0^{(1)}(\kappa |x-y|), & x \neq y, \quad m = 2, \\ \dfrac{e^{i\kappa|x-y|}}{4\pi |x-y|}, & x \neq y, \quad m = 3, \end{cases} \tag{1.4}$$

where $H_0^{(1)}$ is the Hankel function of order zero and of the first kind. Given a function φ, the integrals

$$u(x) := \int_\Gamma \varphi(y)\, \Phi(x,y)\, ds(y), \quad x \in \mathbb{R}^m \setminus \Gamma, \tag{1.5}$$

and

$$v(x) := \int_\Gamma \varphi(y) \frac{\partial \Phi(x,y)}{\partial n(y)}\, ds(y), \quad x \in \mathbb{R}^m \setminus \Gamma, \tag{1.6}$$

are called *acoustic single-layer* and *acoustic double-layer potentials* with density φ. Obviously, these potentials represent solutions to the Helmholtz equation (1.1) and satisfy the radiation condition (1.3). Their behaviour at the boundary is described by the following *jump-relations*.

Let Γ be of class C^2 and let φ be continuous. Then, the single-layer potential u with density φ is continuous throughout $\mathbb{R}^m$. On the boundary, there holds

$$u(x) = \int_\Gamma \varphi(y) \Phi(x,y) ds(y), \qquad x \in \Gamma, \tag{1.7}$$

and

$$\frac{\partial u_\pm}{\partial n}(x) = \int_\Gamma \varphi(y) \frac{\partial \Phi(x,y)}{\partial n(x)}\, ds(y) \mp \frac{1}{2}\varphi(x), \qquad x \in \Gamma, \tag{1.8}$$

where

$$\frac{\partial u_\pm}{\partial n}(x) := \lim_{h \to +0} (n(x), \operatorname{grad} u(x \pm h\, n(x)))$$

is to be understood in the sense of uniform convergence on Γ, and where the integrals exist as improper integrals. The double-layer potential v with density φ can be continuously extended from D_+ to $\overline{D}_+$ and from D_- to $\overline{D}_-$, with limiting values

$$v_\pm(x) = \int_\Gamma \varphi(y) \frac{\partial \Phi(x,y)}{\partial n(y)}\, ds(y) \pm \frac{1}{2}\varphi(x), \qquad x \in \Gamma, \tag{1.9}$$

where

$$v_\pm(x) := \lim_{h \to +0} v(x \pm h\, n(x)),$$

and where the integral exists as an improper integral. Furthermore,

$$\lim_{h\to+0}\left\{\frac{\partial v}{\partial n}(x+h\,n(x))-\frac{\partial v}{\partial n}(x-h\,n(x))\right\}=0,\qquad x\in\Gamma,\tag{1.10}$$

uniformly on Γ. For a proof we refer to [1].

We seek the solution to the Dirichlet problem in the form of a combined acoustic double- and single-layer potential

$$u(x)=\int_\Gamma\left\{\frac{\partial\Phi(x,y)}{\partial n(y)}-i\eta\,\Phi(x,y)\right\}\varphi(y)\,ds(y),\qquad x\in\mathbb{R}^m\setminus\Gamma,\tag{1.11}$$

with some positive coupling parameter η. Then, from (1.9) and (1.7), we see that (1.11) solves the exterior Dirichlet problem, provided the density is a solution of the integral equation

$$\varphi+K\varphi-i\eta S\varphi=2f,\tag{1.12}$$

where $K,S:C(\Gamma)\to C(\Gamma)$ denote the integral operators defined by

$$(K\varphi)(x):=2\int_\Gamma\frac{\partial\Phi(x,y)}{\partial n(y)}\varphi(y)\,ds(y),\qquad x\in\Gamma,\tag{1.13}$$

and

$$(S\varphi)(x):=2\int_\Gamma\Phi(x,y)\,\varphi(y)\,ds(y),\qquad x\in\Gamma.\tag{1.14}$$

These integral operators are compact since they can be shown to have weakly singular kernels. Therefore, the existence of a solution to (1.12) can be established by the Riesz theory for equations of the second kind with a compact operator. Let φ be a solution to the homogeneous form of (1.12). Then the potential u given by (1.11) satisfies the homogeneous boundary condition $u_+=0$ on Γ, whence by the uniqueness for the exterior Dirichlet problem $u=0$ in D_+ follows. Now the jump-relations (1.7) to (1.10) yield

$$-u_-=\varphi\quad\text{and}\quad-\frac{\partial u_-}{\partial n}=i\eta\,\varphi\qquad\text{on }\Gamma.$$

Hence, using Green's integral theorem and (1.1), we obtain

$$i\eta\int_\Gamma|\varphi|^2ds=\int_\Gamma\bar{u}_-\frac{\partial u_-}{\partial n}\,ds=\int_{D_-}\left\{|\operatorname{grad}u|^2-\kappa^2|u|^2\right\}dx.$$

Taking the imaginary part of the last equation ensures $\varphi=0$. Thus we have established uniqueness for the integral equation (1.12), whence existence of a solution to the inhomogeneous equation follows from the Riesz theory.

Note that for $\eta=0$, the integral equation (1.12) becomes non-unique if κ^2 is an eigenvalue of the interior Neumann problem in D_-, a so-called irregular wave number. The approach (1.11) was introduced independently by Leis [2], Brakhage and Werner [3] and Panich [4] in order to remedy this non-uniqueness deficiency of the classical double-layer approach. For an investigation of the proper choice of the coupling parameter η with respect to the condition of the integral equation (1.12) we refer to [5].

In addition to transforming the boundary value problem to integral equations by the layer approach, it is also possible to obtain integral equations using the *direct method* based on the *Green's formula* for solutions to the Helmholtz equation. Let u be a solution to the Helmholtz equation (1.1) which is continuously differentiable up to the boundary Γ and which satisfies the radiation condition (1.3). Then (see [1, Theorem 3.3])

$$u(x)=\int_\Gamma\left\{u(y)\frac{\partial\Phi(x,y)}{\partial n(y)}-\frac{\partial u}{\partial n}(y)\,\Phi(x,y)\right\}ds(y),\qquad x\in D_+.\tag{1.15}$$

From (1.15), letting x tend to the boundary and using the jump-relations (1.7) and (1.9), we see that

$$u - Ku + S\frac{\partial u}{\partial n} = 0. \tag{1.16}$$

Taking the normal derivative on the boundary and using (1.8) shows that

$$\frac{\partial u}{\partial n} - Tu + K'\frac{\partial u}{\partial n} = 0, \tag{1.17}$$

where $K' : C(\Gamma) \to C(\Gamma)$ denotes the adjoint of K, given by

$$(K'\psi)(x) := 2\int_\Gamma \frac{\partial \Phi(x,y)}{\partial n(x)}\psi(y)\,ds(y), \qquad x \in \Gamma, \tag{1.18}$$

and where T stands for the *hypersingular* operator given by the normal derivative of the double-layer potential

$$(T\psi)(x) := 2\frac{\partial}{\partial n(x)}\int_\Gamma \frac{\partial \Phi(x,y)}{\partial n(y)}\psi(y)\,ds(y), \qquad x \in \Gamma. \tag{1.19}$$

Observe that the normal derivative of the double-layer potential exists, since by assumption the normal derivative of u exists.

Now let u be the solution to the exterior Dirichlet problem with boundary condition $u = f$ on Γ. Then, by linearly combining (1.16) and (1.17), we obtain the integral equation

$$\psi + K'\psi - i\eta S\psi = Tf - i\eta(Kf - f) \tag{1.20}$$

for the unknown normal derivative $\psi := \partial u/\partial n$ on Γ. This equation is the adjoint of equation (1.12) obtained from the layer approach. Hence, by the Fredholm alternative, equation (1.20) is uniquely solvable. Having solved the integral equation for ψ, the solution of the boundary value problem is given through the Green's formula (1.15).

We wish to point out that it is also possible to derive integral equations of the first kind by either the layer or the direct approach. But since, in general, the treatment of integral equations of the first kind, theoretically and numerically, is more delicate, our advice is to rely on equations of the second kind whenever they are available.

2. PARAMETRIZATION OF THE INTEGRAL EQUATIONS

We continue by describing the parametrization of the integral equation (1.12) in the two-dimensional case. We assume that the boundary curve Γ is analytic, with a regular parametric representation of the form

$$x(t) = (x_1(t), x_2(t)), \qquad 0 \le t \le 2\pi, \tag{2.1}$$

in counter-clockwise orientation satisfying $[x_1'(t)]^2 + [x_2'(t)]^2 > 0$, for all $0 \le t \le 2\pi$. Then, by straightforward calculations, we transform (1.12) into the parametric form

$$\psi(t) - \int_0^{2\pi} \{L(t,\tau) + i\eta M(t,\tau)\}\psi(\tau)\,d\tau = 2g(t), \qquad 0 \le t \le 2\pi, \tag{2.2}$$

where we have set $\psi(t) := \varphi(x(t))$ and $g(t) := f(x(t))$, and where the kernels are given by

$$L(t,\tau) := \frac{i\kappa}{2}\{x_2'(\tau)[x_1(\tau) - x_1(t)] - x_1'(\tau)[x_2(\tau) - x_2(t)]\}\frac{H_1^{(1)}(\kappa r(t,\tau))}{r(t,\tau)},$$

and

$$M(t,\tau) := \frac{i}{2}H_0^{(1)}(\kappa r(t,\tau))\{[x_1'(\tau)]^2 + [x_2'(\tau)]^2\}^{1/2},$$

for $t \neq \tau$. Here, we have set

$$r(t,\tau) := \{[x_1(t) - x_1(\tau)]^2 + [x_2(t) - x_2(\tau)]^2\}^{1/2},$$

and $H_1^{(1)}(z) = -dH_0^{(1)}(z)/dz$ denotes the Hankel function of order one and of the first kind. We decompose the fundamental solution $H_0^{(1)} = J_0 + i\,N_0$, and use the power series

$$J_0(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{z}{2}\right)^{2k}, \tag{2.3}$$

for the Bessel function of order zero and

$$N_0(z) = \frac{2}{\pi} \left(\ln \frac{z}{2} + C\right) J_0(z) + \sum_{k=1}^{\infty} \left\{ \sum_{m=1}^{k} \frac{1}{m} \right\} \frac{(-1)^{k+1}}{(k!)^2} \left(\frac{z}{2}\right)^{2k}, \tag{2.4}$$

for the Neumann function of order zero, with Euler's constant $C = 0.57721\ldots$. From the expansions (2.3) and (2.4) and their term by term derivatives, we see that the kernels L and M have logarithmic singularities. Actually, L turns out to be continuous but its derivatives have logarithmic singularities. Hence, for their proper numerical treatment we split the kernels into

$$L(t,\tau) = L_1(t,\tau) \ln\left(4 \sin^2 \frac{t-\tau}{2}\right) + L_2(t,\tau), \tag{2.5}$$

and

$$M(t,\tau) = M_1(t,\tau) \ln\left(4 \sin^2 \frac{t-\tau}{2}\right) + M_2(t,\tau), \tag{2.6}$$

where

$$\begin{aligned}
L_1(t,\tau) &:= \frac{\kappa}{2\pi} \{x_2'(\tau)\,[x_1(t) - x_1(\tau)] - x_1'(\tau)\,[x_2(t) - x_2(\tau)]\} \frac{J_1(\kappa\, r(t,\tau))}{r(t,\tau)}, \\
L_2(t,\tau) &:= L(t,\tau) - L_1(t,\tau) \ln\left(4 \sin^2 \frac{t-\tau}{2}\right), \\
M_1(t,\tau) &:= -\frac{1}{2\pi} J_0(\kappa\, r(t,\tau))\, \{[x_1'(\tau)]^2 + [x_2'(\tau)]^2\}^{1/2}, \\
M_2(t,\tau) &:= M(t,\tau) - M_1(t,\tau) \ln\left(4 \sin^2 \frac{t-\tau}{2}\right).
\end{aligned}$$

Of course, $J_1(z) = -dJ_0(z)/dz$ denotes the Bessel function of order one. The kernels L_1, L_2, M_1, and M_2 turn out to be analytic. In particular, using the expansions (2.3) and (2.4), we can deduce the diagonal terms

$$L_2(t,t) = L(t,t) = \frac{1}{2\pi} \frac{x_1'(t)\,x_2''(t) - x_2'(t)\,x_1''(t)}{[x_1'(t)]^2 + [x_2'(t)]^2},$$

and

$$M_2(t,t) = \left\{ \frac{i}{2} - \frac{C}{\pi} - \frac{1}{2\pi} \ln \frac{\kappa^2}{4} \{[x_1'(t)]^2 + [x_2'(t)]^2\} \right\} \{[x_1'(t)]^2 + [x_2'(t)]^2\}^{1/2},$$

for $0 \leq t \leq 2\pi$. We note that, despite the continuity of the kernel L, for numerical accuracy it is advantageous to separate the logarithmic part of L, since the derivatives of L fail to be continuous. We also note that L_1, L_2, M_1, and M_2 are continuous if Γ is merely assumed to be C^2.

The integral equation (1.20) can be parametrized analogously. On the right hand side, for the hypersingular operator T defined by (1.19) we make use of the identity

$$\frac{\partial v}{\partial n}(x) = \kappa^2 \int_\Gamma (n(x), n(y))\, \Phi(x,y)\, \varphi(y)\, ds(y) + \frac{d}{ds(x)} \int_\Gamma \Phi(x,y) \frac{d\varphi}{ds}(y)\, ds(y), \qquad x \in \Gamma, \tag{2.7}$$

for the normal derivative of the double-layer potential with density φ. For a derivation of (2.7) we refer to [1, Theorem 2.23] for the three-dimensional case, and to [6, Theorem 7.28] for the two-dimensional case with $\kappa = 0$. Now, up to a factor $\{[x_1'(t)]^2 + [x_2'(t)]^2\}^{-1/2}$, the parametrization of the second term on the right hand side of (2.7) yields an integral of the form

$$\int_0^{2\pi} N(t,\tau)\,\psi'(\tau)\,d\tau, \qquad 0 \le t \le 2\pi, \tag{2.8}$$

with the kernel

$$N(t,\tau) := \frac{i\kappa}{2}\,\{x_1'(t)\,[x_1(\tau) - x_1(t)] + x_2'(t)\,[x_2(\tau) - x_2(t)]\}\,\frac{H_1^{(1)}(\kappa\, r(t,\tau))}{r(t,\tau)},$$

for $t \neq \tau$. Here, in addition to the logarithmic singularity, we also encounter a Cauchy type singularity. Therefore, we split

$$N(t,\tau) = \frac{1}{2\pi}\cot\frac{\tau - t}{2} + N_1(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right) + N_2(t,\tau), \tag{2.9}$$

where

$$N_1(t,\tau) := \frac{\kappa}{2\pi}\,\{x_1'(t)\,[x_1(t) - x_1(\tau)] + x_2'(t)\,[x_2(t) - x_2(\tau)]\}\,\frac{J_1(\kappa\, r(t,\tau))}{r(t,\tau)},$$

$$N_2(t,\tau) := N(t,\tau) - \frac{1}{2\pi}\cot\frac{\tau - t}{2} - N_1(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right).$$

Again, the kernels N_1 and N_2 are analytic with the diagonal term

$$N_2(t,t) - \frac{1}{2\pi}\,\frac{x_1'(t)\,x_1''(t) + x_2'(t)\,x_2''(t)}{[x_1'(t)]^2 + [x_2'(t)]^2}.$$

Hence, for both the approaches (1.12) and (1.20) to the Dirichlet problem, in principle, we have to numerically solve an integral equation of the form

$$\psi(t) - \int_0^{2\pi} K(t,\tau)\,\psi(\tau)\,d\tau = g(t), \qquad 0 \le t \le 2\pi, \tag{2.10}$$

where the kernel can be split into

$$K(t,\tau) = K_1(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right) + K_2(t,\tau) \tag{2.11}$$

with smooth functions K_1 and K_2, and with a smooth right hand side g.

3. NYSTRÖM'S METHOD

The *Nyström method* for the numerical solution of integral equations of the second kind consists in the straightforward approximation of the integrals by quadrature formulas. In our case, we choose an equidistant set of knots $t_k := \pi k/n$, $k = 0, \ldots, 2n-1$, and use the quadrature rule

$$\int_0^{2\pi} \ln\left(4\sin^2\frac{t-\tau}{2}\right) f(\tau)\,d\tau \approx \sum_{k=0}^{2n-1} R_k^{(n)}(t)\,f(t_k), \qquad 0 \le t \le 2\pi, \tag{3.1}$$

with the quadrature weights given by

$$R_k^{(n)}(t) = -\frac{2\pi}{n}\sum_{m=1}^{n-1}\frac{1}{m}\cos m(t - t_k) - \frac{\pi}{n^2}\cos n(t - t_k), \qquad k = 0, \ldots, 2n-1,$$

and the trapezoidal rule

$$\int_0^{2\pi} f(\tau)\,d\tau \approx \frac{\pi}{n} \sum_{k=0}^{2n-1} f(t_k). \tag{3.2}$$

Both these numerical integration formulas are obtained by replacing the integrand f by its trigonometric interpolation polynomial and then integrating exactly. The quadrature formula (3.1) was first used by Martensen [7] and Kussmaul [8]. Provided f is analytic, according to derivative-free error estimates for the remainder term in trigonometric interpolation for periodic analytic functions (see [6,9]), the errors for the quadrature rules (3.1) and (3.2) decrease at least exponentially when the number $2n$ of knots is increased. More precisely, the error is of order $O(\exp(-n\sigma))$, where σ denotes half of the width of a parallel strip in the complex plane into which the real analytic function f can be holomorphically extended.

In the Nyström method, the integral equation (2.10) is replaced by the approximating equation

$$\psi^{(n)}(t) - \sum_{k=0}^{2n-1} \left\{ R_k^{(n)}(t)\, K_1(t,t_k) + \frac{\pi}{n} K_2(t,t_k) \right\} \psi^{(n)}(t_k) = g(t), \tag{3.3}$$

for $0 \le t \le 2\pi$. Equation (3.3) is obtained from (2.10) by applying the quadrature rule (3.1) to $f = K_1(t,.)\,\psi$ and (3.2) to $f = K_2(t,.)\,\psi$. Now, solving (3.3) reduces to solving a finite dimensional linear system. For any solution of (3.3), the values $\psi_i^{(n)} = \psi^{(n)}(t_i)$, $i = 0,\dots,2n-1$, at the quadrature points trivially satisfy the linear system

$$\psi_i^{(n)} - \sum_{k=0}^{2n-1} \left\{ R_{|i-k|}^{(n)}\, K_1(t_i,t_k) + \frac{\pi}{n} K_2(t_i,t_k) \right\} \psi_k^{(n)} = g(t_i), \tag{3.4}$$

for $i = 0,\dots,2n-1$, where

$$R_k^{(n)} := R_k^{(n)}(0) = -\frac{2\pi}{n} \sum_{m=1}^{n-1} \frac{1}{m} \cos\frac{mk\pi}{n} - \frac{(-1)^k \pi}{n^2}, \qquad k = 0,\dots,2n-1.$$

And conversely, given a solution $\psi_i^{(n)}$, $i = 0,\dots,2n-1$, of the system (3.4), then the function $\psi^{(n)}$ defined by

$$\psi^{(n)}(t) := \sum_{k=0}^{2n-1} \left\{ R_k^{(n)}(t)\, K_1(t,t_k) + \frac{\pi}{n} K_2(t,t_k) \right\} \psi_k^{(n)} + g(t), \tag{3.5}$$

for $0 \le t \le 2\pi$, is readily seen to satisfy the approximating equation (3.3). The formula (3.5) may be viewed as a natural interpolation of the values $\psi_i^{(n)}$, $i = 0,\dots,2n-1$, at the quadrature points to obtain the approximating function $\psi^{(n)}$, and goes back to Nyström [10].

For the solution of the linear system (3.4) we recommend the use of the fast iterative two-grid or multi-grid methods, as described in [11].

Provided the integral equation (2.10) itself is uniquely solvable and the kernels K_1 and K_2, and the right hand side g are continuous, a rather involved error analysis based on Anselone's [12] concept of *collectively compact operators*, shows that:

1. The approximating linear system (3.4), i.e., the approximating equation (3.3), is uniquely solvable for all sufficiently large n.
2. For $n \to \infty$, the approximate solutions $\psi^{(n)}$ converge uniformly to the solution ψ of the integral equation.
3. The convergence order of the quadrature errors for (3.1) and (3.2) carries over to the error $\psi^{(n)} - \psi$.

The latter, in particular, means that in the case of analytic kernels K_1 and K_2, and analytic right hand sides g, the approximation error decreases exponentially, i.e., doubling the number $2n$ of knots will double the number of correct digits in the approximate solution. For details of this error analysis, we refer to [6].

For the evaluation of the right hand side of (1.20) via (2.7)–(2.9), we suggest to use the quadrature rule due to Garrick [13] and Wittich [14] for the Cauchy integral

$$\int_0^{2\pi} \cot\frac{t}{2}\, f(t)\, dt \approx \frac{2\pi}{n} \sum_{k=1}^{n} \cot\frac{t_{2k-1}}{2}\, f(t_{2k-1}),$$

and trigonometric differentiation, i.e., to replace the derivative of f by the derivative of the trigonometric interpolation polynomial

$$\frac{1}{2n} \sum_{m=0}^{2n-1} \sin n(t-t_m) \cot\frac{t-t_m}{2}\, f(t_m),$$

that is,

$$f'(t_k) \approx \frac{1}{2} \sum_{\substack{m=0 \\ m\neq k}}^{2n-1} (-1)^{k-m} \cot\frac{t_k - t_m}{2}\, f(t_m), \qquad k = 0, \ldots, 2n-1.$$

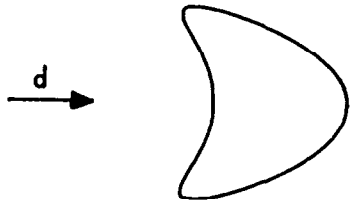

Figure 1.

For a numerical example we consider the scattering of an incident plane wave u^i at a sound-soft non-convex kite-shaped domain with boundary Γ described by the parametric representation (see Figure 1),

$$x(t) = (\cos t + 0.65\cos 2t - 0.65\ ,\ 1.5\sin t), \qquad 0 \le t \le 2\pi.$$

The incident wave is given by $u^i(x) = e^{i\kappa(d,x)}$, where d denotes a unit vector giving the direction of propagation. For the scattered wave u we have to solve an exterior Dirichlet problem with boundary values $f = -u^i$ on Γ. The *far-field pattern* u_∞ is defined by the asymptotic behaviour of the scattered wave

$$u(x) = \frac{e^{i\kappa|x|}}{\sqrt{|x|}} \left\{ u_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right) \right\}, \qquad |x| \to \infty,$$

uniformly in all directions $\hat{x} = x/|x|$. From the asymptotic behaviour

$$H_n^{(1)}(z) = \sqrt{\frac{2}{\pi z}}\, e^{i(z - n\frac{\pi}{2} - \frac{\pi}{4})} \left\{ 1 + O\left(\frac{1}{z}\right) \right\}, \qquad z \to \infty,$$

for the Hankel functions of order n and of the first kind, we see that the far-field pattern of the combined potential (1.11) is given by

$$u_\infty(\hat{x}) = \gamma \int_\Gamma \{\kappa(n(y), \hat{x}) + \eta\}\, e^{-i\kappa(\hat{x},y)}\, \varphi(y)\, ds(y), \tag{3.6}$$

where $\gamma = e^{-i\frac{\pi}{4}}/\sqrt{8\pi\kappa}$. After solving the integral equation (1.12) numerically, the integral (3.6) is again evaluated by the trapezoidal rule. Table 1 gives some approximate values for the far-field pattern $u_\infty(d)$ and $u_\infty(-d)$ in the forward direction d and the backward direction $-d$. The direction d of the incident wave is $d = (1, 0)$ and, as recommended in [5], the coupling parameter is $\eta = \kappa$. Clearly, the exponential convergence is exhibited.

Table 1. Numerical results for Nyström's method.

	n	$\operatorname{Re} u_\infty(d)$	$\operatorname{Im} u_\infty(d)$	$\operatorname{Re} u_\infty(-d)$	$\operatorname{Im} u_\infty(-d)$
$\kappa = 1$	8	-1.62642413	0.60292714	1.39015283	0.09425130
	16	-1.62745909	0.60222343	1.39696610	0.09499454
	32	-1.62745750	0.60222591	1.39694488	0.09499635
	64	-1.62745750	0.60222591	1.39694488	0.09499635
$\kappa = 5$	8	-2.30969119	1.52696566	-0.30941096	0.11503232
	16	-2.46524869	1.67777368	-0.19932343	0.06213859
	32	-2.47554379	1.68747937	-0.19945788	0.06015893
	64	-2.47554380	1.68747937	-0.19945787	0.06015893

4. COLLOCATION METHOD

In the collocation method we approximate the solution to the integral equation (2.10) by a trigonometric polynomial of the form

$$\psi^{(n)}(t) = \sum_{k=0}^{n} a_k \cos kt + \sum_{k=1}^{n-1} b_k \sin kt, \tag{4.1}$$

by requiring that the integral equation be satisfied at the collocation points $t_i := \pi i/n$, $i = 0, \ldots, 2n-1$, that is,

$$\psi^{(n)}(t_i) - \int_0^{2\pi} K(t_i, \tau)\, \psi^{(n)}(\tau)\, d\tau = g(t_i), \qquad i = 0, \ldots, 2n-1. \tag{4.2}$$

Plugging (4.1) into (4.2), in principle, we obtain a linear system for the coefficients a_k and b_k of the approximating trigonometric polynomial. But it is more efficient to replace the representation (4.1) by using the Lagrange basis $L_0, \ldots, L_{2n-1}$ defined by the interpolation property

$$L_k(t_i) = \delta_{ik}, \qquad i, k = 0, \ldots, 2n-1.$$

This basis can be given explicitly by

$$L_k(t) = \frac{1}{2n} \sin n(t - t_k) \cot \frac{t - t_k}{2}, \qquad t \neq t_k, \tag{4.3}$$

for $k = 0, \ldots, 2n-1$. Now we write

$$\psi^{(n)} = \sum_{k=0}^{2n-1} \gamma_k^{(n)} L_k. \tag{4.4}$$

Then, of course, $\gamma_i^{(n)} = \psi^{(n)}(t_i)$, $i = 0, \ldots, 2n-1$, and solving the equation (4.2) reduces to solving the linear system

$$\gamma_i^{(n)} - \sum_{k=0}^{2n-1} \gamma_k^{(n)} \int_0^{2\pi} K(t_i, \tau)\, L_k(\tau)\, d\tau = g(t_i), \qquad i = 0, \ldots, 2n-1. \tag{4.5}$$

The collocation method can be considered as a special case of a *projection method* with the projection operator generated by the interpolation. Hence, the general error analysis for projection methods is applicable. Provided the integral equation (2.10) itself is uniquely solvable, and K_1 is twice, and K_2 and g are once continuously differentiable, then

1. The approximating linear system (4.5), i.e., the approximating equation (4.2), is uniquely solvable for all sufficiently large n.

2. For $n \to \infty$, the approximate solutions $\psi^{(n)}$ converge uniformly to the solution ψ of the integral equation.
3. The convergence order of the error $\psi^{(n)} - \psi$ coincides with the convergence order of the error $\psi_{int}^{(n)} - \psi$ for the trigonometric interpolation $\psi_{int}^{(n)}$ of the exact solution ψ.

The latter, in particular, means that in the case of analytic kernels K_1 and K_2, and analytic right hand sides g, the approximation error again decreases exponentially. For details, we refer to [6].

Recalling the form (2.11) of the kernel K, we observe that for the evaluation of the matrix elements in (4.5), we need quadrature formulas for integrals of the form

$$\int_0^{2\pi} K_1(t_i, \tau) \ln\left(4 \sin^2 \frac{t_i - \tau}{2}\right) L_k(\tau)\, d\tau$$

and

$$\int_0^{2\pi} K_2(t_i, \tau)\, L_k(\tau)\, d\tau.$$

To be consistent with our approximations, we replace $K_1(t_i, \cdot)$ and $K_2(t_i, \cdot)$ by their trigonometric interpolation, i.e., we approximate

$$\begin{aligned} &\int_0^{2\pi} K_1(t_i, \tau) \ln\left(4 \sin^2 \frac{t_i - \tau}{2}\right) L_k(\tau)\, d\tau \\ &\approx \sum_{m=0}^{2n-1} K_1(t_i, t_m) \int_0^{2\pi} \ln\left(4 \sin^2 \frac{t_i - \tau}{2}\right) L_k(\tau)\, L_m(\tau)\, d\tau, \end{aligned} \tag{4.6}$$

and

$$\int_0^{2\pi} K_2(t_i, \tau)\, L_k(\tau)\, d\tau \approx \sum_{m=0}^{2n-1} K_2(t_i, t_m) \int_0^{2\pi} L_k(\tau)\, L_m(\tau)\, d\tau, \tag{4.7}$$

for $i, k = 0, \ldots, 2n-1$. Using (4.3), elementary integrations yield

$$\int_0^{2\pi} L_k(\tau)\, L_m(\tau)\, d\tau = \frac{\pi}{n}\, \delta_{km} - (-1)^{k-m}\, \frac{\pi}{4n^2}$$

for the weights in (4.7). A practical numerical evaluation of the weights in (4.6) rests on the observation that the quadrature rule (3.1), with the number of grid points $2n$ doubled to $4n$, integrates trigonometric polynomials of degree less than or equal to $2n$ exactly provided the coefficient of the term $\sin 2nt$ vanishes. Therefore, we have

$$\int_0^{2\pi} \ln\left(4 \sin^2 \frac{\tau}{2}\right) L_k(\tau)\, L_m(\tau)\, d\tau = \sum_{j=0}^{4n-1} R_j^{(2n)}\, L_k(t_j^{(2n)})\, L_m(t_j^{(2n)}).$$

In the case of analytic kernels K_1 and K_2, we have an exponentially decreasing error in the last two quadrature formulas. It can be shown that, with these approximations for the matrix elements in (4.5), the total error is still exponentially decreasing for analytic kernels K_1 and K_2, and analytic right hand sides g. For details, we again refer to [6].

Our treatment differs from the collocation method described in [15] through the use of the Lagrange basis and an approximation of the matrix elements which preserves the exponential convergence.

In principle, using higher order trigonometric interpolation for $K_1(t_i, .)$ and $K_2(t_i, .)$, it would be possible to evaluate the matrix elements more accurately. In general, this does not pay off since it will only yield marginal improvements in the accuracy of the approximate solution. Instead of investing computing efforts in a highly accurate evaluation of the matrix elements, it is more efficient to increase the number of collocation points, i.e., to increase the size of the linear system.

Because the approximation order for Nyström's method and for the collocation method is the same, Nyström's method is preferable since it requires the least computational effort for

evaluating the matrix elements. In addition, the Nyström method is generically stable in the sense that it preserves the condition of the integral equation, whereas in the collocation method this condition can be disturbed by a poor choice of the basis.

The following Table 2 gives the numerical results for the example introduced at the end of Section 3. The significant figures coincide with those of Table 1, i.e., the additional computational effort for the collocation method is ineffective.

Table 2. Numerical results for the collocation method.

	n	$\mathrm{Re}\, u_\infty(d)$	$\mathrm{Im}\, u_\infty(d)$	$\mathrm{Re}\, u_\infty(-d)$	$\mathrm{Im}\, u_\infty(-d)$
$\kappa = 1$	8	-1.62647231	0.60291074	1.39019406	0.09416353
	16	-1.62745916	0.60222339	1.39696613	0.09499463
	32	-1.62745750	0.60222591	1.39694488	0.09499635
	64	-1.62745750	0.60222591	1.39694488	0.09499635
$\kappa = 5$	8	-2.58411979	1.26947888	-0.70515669	-0.05840491
	16	-2.47551177	1.68663257	-0.19871124	0.06097800
	32	-2.47554382	1.68747936	-0.19945782	0.06015891
	64	-2.47554380	1.68747937	-0.19945787	0.06015893

5. GALERKIN METHOD

In the *Galerkin method*, the solution to the integral equation (2.10) is approximated through a trigonometric polynomial $\psi^{(n)}$ of the form (4.1), by requiring that

$$\int_0^{2\pi} \left\{ \psi^{(n)}(t) - \int_0^{2\pi} K(t,\tau)\, \psi^{(n)}(\tau)\, d\tau \right\} \varphi(t)\, dt = \int_0^{2\pi} g(t)\, \varphi(t)\, dt, \tag{5.1}$$

for all trigonometric polynomials φ of the form (4.1). Using the representation (4.4), the Galerkin equation (5.1) reduces to the linear system

$$\sum_{k=0}^{2n-1} \left\{ \gamma_k^{(n)} \int_0^{2\pi} L_i(t)\, L_k(t)\, dt - \int_0^{2\pi} \int_0^{2\pi} K(t,\tau)\, L_i(t)\, L_k(\tau)\, d\tau\, dt \right\} = \int_0^{2\pi} g(t)\, L_i(t)\, dt, \tag{5.2}$$

for $i = 0, \ldots, 2n-1$. The Galerkin method can be considered as a special case of a projection method with the orthogonal projection operator. Provided the integral equation (2.10) itself is uniquely solvable, and K_1, K_2, and g are continuous, then from the general error analysis for projection methods, it follows that

1. The approximating linear system (5.2), i.e., the approximating equation (5.1), is uniquely solvable for all sufficiently large n.
2. For $n \to \infty$, the approximate solutions $\psi^{(n)}$ converge in the mean square to the solution ψ of the integral equation.
3. The convergence order of the error $\psi^{(n)} - \psi$ coincides with the convergence order of the error $\psi_{best}^{(n)} - \psi$ for the best approximation in the mean square $\psi_{best}^{(n)}$ of the exact solution ψ with respect to the trigonometric polynomials .

Recalling the form (2.11) of the kernel K, we observe that for the evaluation of the matrix elements in (5.2) we need quadrature formulas for integrals of the form

$$\int_0^{2\pi} \int_0^{2\pi} K_1(t,\tau) \ln\left(4 \sin^2 \frac{t-\tau}{2}\right) L_i(t)\, L_k(\tau)\, d\tau\, dt,$$

and

$$\int_0^{2\pi} \int_0^{2\pi} K_2(t,\tau)\, L_i(t)\, L_k(\tau)\, d\tau\, dt.$$

If we pursue the same idea as in the collocation method and use interpolatory quadratures with respect to both variables, it turns out that we solve the same approximate finite dimensional equation on the space of trigonometric polynomials as in the collocation method. Therefore, the numerical results will be the same. Only the method of solution for this equation is different: in the collocation method we equate trigonometric polynomials by requiring them to coincide at the interpolation grid, in the Galerkin method we require equality of scalar products with respect to the basis given by (4.3). Therefore, this coincidence is indicated by denoting such a procedure as a *Galerkin collocation method*. Hence, we do not need to repeat the numerical calculations for our example of the two previous sections.

6. DOMAINS WITH CORNERS

In this final section we wish to cover the important case of domains with corners. Without loss of generalization, we confine our presentation to a boundary curve Γ with one corner at the point x_0. The extension to a finite number of corners is straightforward. We assume $\Gamma \setminus x_0$ to be analytic and choose a parametric representation of the form (2.1) such that the corner x_0 corresponds to the parameter $t = 0$.

Due to the jump-relations for the double-layer potential at a corner, the integral equation (1.12) has to be modified into the form

$$\alpha\varphi + K\varphi - i\eta S\varphi = 2f, \tag{6.1}$$

where $\alpha(x) = 1$, for $x \neq x_0$ and $\alpha(x_0) = \gamma_0/\pi$. Here, by γ_0 we denote the interior angle at the corner x_0. The integral operator in (6.1) no longer remains compact. However, following the classical ideas of Radon [16], it can be shown that it can be written as the sum of an operator, with norm less than one and a compact operator, provided $0 < \gamma_0 < 2\pi$. Hence, the existence of a continuous solution can still be established. For details, we refer to Ruland [17].

But this solution will have singularities in the derivatives at the corner. Due to the singularities of solutions to elliptic boundary value problems in domains with corners (see [18]), after parametrizing, we expect, for the derivatives of the solution ψ, the asymptotic behaviour

$$\psi^{(m)}(t) = O(t^{\beta-m}), \quad \psi^{(m)}(2\pi - t) = O(t^{\beta-m}), \qquad t \to 0, \tag{6.2}$$

for $m = 0, 1, \ldots$, with some $\beta > 0$. To take proper care of this corner singularity, we have to replace our equidistant mesh by a graded mesh.

We base our grading upon the idea of substituting a new variable in such a way that all the derivatives of the new integrand vanish at the endpoints $t = 0$ and $t = 2\pi$. Following Sag and Szekeres [19], we define

$$w(s) := 2\pi \frac{\exp\left(-\dfrac{2\pi}{s}\right)}{\exp\left(-\dfrac{2\pi}{s}\right) + \exp\left(-\dfrac{2\pi}{2\pi - s}\right)}, \qquad 0 \leq s \leq 2\pi. \tag{6.3}$$

The function w is monotonically increasing and performs a one-to-one transformation of $[0, 2\pi]$ onto itself. Consequently,

$$\int_0^{2\pi} g(t)\, dt = \int_0^{2\pi} h(s)\, ds, \tag{6.4}$$

where

$$h(s) := w'(s)\, g(w(s)), \qquad 0 \leq s \leq 2\pi. \tag{6.5}$$

Applying the trapezoidal rule to the transformed integral now yields the quadrature formula

$$\int_0^{2\pi} g(t)\, dt \approx \frac{\pi}{n} \sum_{k=1}^{2n-1} a_k\, g(s_k), \tag{6.6}$$

with the weights and mesh points given by

$$a_k = w'\left(\frac{k\pi}{n}\right), \quad s_k = w\left(\frac{k\pi}{n}\right), \qquad k = 1, \ldots, 2n-1.$$

Assume that g is infinitely differentiable for $0 < t < 2\pi$ and has a behaviour of the form (6.2) at the endpoints $t = 0$ and $t = 2\pi$. Then, it can be shown that h is infinitely differentiable for $0 \leq s \leq 2\pi$, with all derivatives vanishing at the endpoints

$$h^{(m)}(0) = h^{(m)}(2\pi) = 0, \qquad m = 0, 1, \ldots.$$

Hence, from the Euler-MacLaurin expansion, applied to $\int_0^{2\pi} h(s)ds$, for the remainder term $E^{(n)}(g)$ of the quadrature (6.6), we obtain a sequence of estimates

$$|E^{(n)}(g)| \leq \frac{C(q)}{n^{2q+1}}, \qquad q = 0, 1, \ldots. \tag{6.7}$$

These estimates indicate that the error decreases faster than all powers of $1/n$, i.e., the convergence may be considered as almost exponential. Of course, we have to be aware of the fact that, in general, the order constants $C(q)$, representing the $(2q+1)$-th derivative of h will be increasing with q.

For the numerical solution of the integral equation (2.10) by Nyström's method on the graded mesh, we also have to take into account the logarithmic singularity. We transform $t = w(s)$ and $\tau = w(\sigma)$ to obtain

$$\int_0^{2\pi} K(t,\tau)\,\psi(\tau)\,d\tau = \int_0^{2\pi} K(w(s), w(\sigma))\; w'(\sigma)\,\psi(w(\sigma))\,d\sigma,$$

and then split

$$K(w(s), w(\sigma)) = \tilde{K}_1(s,\sigma)\ln\left(4\sin^2\frac{s-\sigma}{2}\right) + \tilde{K}_2(s,\sigma).$$

This decomposition is related to (2.11) by

$$\tilde{K}_1(s,\sigma) = K_1(w(s), w(\sigma)),$$

and

$$\tilde{K}_2(s,\sigma) = K(w(s), w(\sigma)) - \tilde{K}_1(s,\sigma)\ln\left(4\sin^2\frac{s-\sigma}{2}\right), \qquad s \neq \sigma,$$

with diagonal term

$$\begin{aligned}\tilde{K}_2(s,s) &= \lim_{\sigma\to s}\left[K(w(s), w(\sigma)) - \tilde{K}_1(s,\sigma)\ln\left(4\sin^2\frac{s-\sigma}{2}\right)\right] \\ &= K_2(w(s), w(s)) + 2\ln w'(s)\, K_1(w(s), w(s)).\end{aligned}$$

Recall that

$$K_2(s,s) = \lim_{\sigma\to s}\left[K(s,\sigma) - K_1(s,\sigma)\ln\left(4\sin^2\frac{s-\sigma}{2}\right)\right].$$

Now, proceeding as in the derivation of (3.4), for the approximate values $\psi_i^{(n)} = \psi^{(n)}(s_i)$ we arrive at the linear system

$$\psi_i^{(n)} - \sum_{k=1}^{2n-1}\left\{R_{|i-k|}^{(n)}\,\tilde{K}_1(s_i, s_k) + \frac{\pi}{n}\,\tilde{K}_2(s_i, s_k)\right\} a_k\,\psi_k^{(n)} = g(s_i), \tag{6.8}$$

for $i = 1, \ldots, 2n-1$. A rigorous error analysis carrying over the error behaviour (6.7) to the approximate solution of the integral equation obtained from (6.8) is in preparation and will be published in due course.

Note that we also can obtain equation (6.8) by changing the parametric representation of the boundary Γ, through the substitution $t = w(s)$, and then applying (3.4) in the transformed parametrization.

Instead of (6.3) we may use other transformations with the same feature of smoothing out the singularities by zeros of infinite order at the endpoints, for example the IMT quadrature rule suggested by Iri, Moriguti and Takesawa [20].

In actual numerical calculations, due to the growth of the order constants $C(q)$, the convergence order with large q will only be active for very large n. Hence, to avoid overgrading, it will be sufficient to use a transformation with zeros of a moderately high order at the endpoints. Our suggestion is to replace (6.3) by

$$w(s) = 2\pi \frac{[v(s)]^q}{[v(s)]^q + [v(2\pi - s)]^q}, \tag{6.9}$$

where

$$v(s) = \left(\frac{1}{q} - \frac{1}{2}\right)\left(\frac{\pi - s}{\pi}\right)^3 + \frac{1}{q}\frac{s - \pi}{\pi} + \frac{1}{2}.$$

Note that the cubic polynomial v is chosen such that $v(0) = 0$, $v(2\pi) = 1$, and $w'(\pi) = 2$. The latter property ensures, roughly speaking, that one half of the grid points is equally distributed over the total interval, whereas the other half is accumulated towards the two end points.

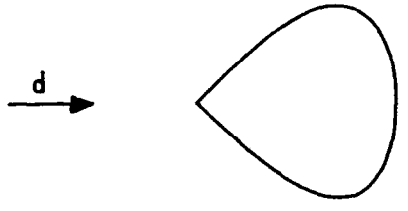

Figure 2.

In our final numerical example, we used the substitution (6.9) with order $q = 8$. We consider a drop-shaped domain (see Figure 2) with the boundary curve Γ given by the parametric representation

$$x(t) = (2\sin\frac{t}{2}, \sin t), \qquad 0 \le t \le 2\pi.$$

It has a corner at $t = 0$ with interior angle $\gamma_0 = \pi/2$. The direction d of the incoming plane wave and the coupling parameter η are chosen as in the example at the end of Section 3. Clearly, Table 3 exhibits the fast convergence of the method.

Table 3. Nyström's method for a domain with corner.

	n	$\mathrm{Re}\, u_\infty(d)$	$\mathrm{Im}\, u_\infty(d)$	$\mathrm{Re}\, u_\infty(-d)$	$\mathrm{Im}\, u_\infty(-d)$
$\kappa = 1$	16	-1.28550023	0.30678332	-0.53079281	-0.41077797
	32	-1.28549379	0.30686386	-0.53022712	-0.41095772
	64	-1.28549353	0.30686622	-0.53021063	-0.41096352
	128	-1.28549352	0.30686627	-0.53021026	-0.41096365
$\kappa = 5$	16	-1.73778683	1.07779542	-0.18211727	-0.20565301
	32	-1.74656244	1.07565779	-0.19431905	-0.19452813
	64	-1.74656303	1.07565737	-0.19429717	-0.19453361
	128	-1.74656304	1.07565737	-0.19429668	-0.19453373

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