

Corollary 5.12 Under the assumptions of Theorem 5.6 the index is stable with respect to compact perturbations, i.e., for compact operators C with a compact adjoint C' we have

$$\text{ind}(K + C) = \text{ind } K.$$

Proof. Let K and K' be adjoint operators with adjoint regularizers R and R' . Then, since $RK = I - A$, where A is compact, Theorems 5.9 and 5.11 imply that

$$\text{ind } R + \text{ind } K = \text{ind } RK = \text{ind } (I - A) = 0,$$

i.e., $\text{ind } K = -\text{ind } R$. For a compact operator C the operator R also regularizes $K + C$ and the operator R' regularizes $K' + C'$. Therefore

$$\text{ind}(K + C) = -\text{ind } R = \text{ind } K,$$

and the proof is complete. \square

For the history of the development of the notion of the index of an operator we refer to [35].

Of course, this chapter can provide only a first glance into the theory of singular operators. For a detailed study, in the canonical dual system $\langle X, X^* \rangle$, we refer to the monograph by Mikhlin and Prössdorf [124].

Problems

5.1 Show that the transformations of the Volterra integral equation of the first kind (3.6) into the Volterra equations of the second kind (3.7) and (3.8) can be interpreted as regularizations from the left and from the right, respectively.
Hint: Use the space $C^1[a, b]$ of continuously differentiable functions furnished with the norm $\|\varphi\|_1 := \|\varphi'\|_\infty + \|\varphi\|_\infty$.

5.2 Convince yourself where in the proof of Theorem 5.6 use is made of the fact that the operators K and K' possess regularizers from the left and from the right.

5.3 Use Theorem 5.9 for an alternative proof of Theorem 4.15.

5.4 Let X_1, X_2 be Banach spaces, let $K : X_1 \rightarrow X_2$ be a bounded operator, and let $R : X_2 \rightarrow X_1$ be a left (right) regularizer of K . Show that for all operators $C : X_1 \rightarrow X_2$ with $\|C\| < \|R\|$ the operator $K + C$ has a left (right) regularizer.

5.5 Use Problem 5.4 to show that in Banach spaces under the assumptions of Theorem 5.6 the index is stable with respect to small perturbations, i.e., there exists a positive number γ (depending on K and K') such that

$$\text{ind}(K + C) = \text{ind } K$$

for all operators C with adjoint C' satisfying $\max(\|C\|, \|C'\|) < \gamma$ (see [7, 34]).

6 Potential Theory

The solution of boundary value problems for partial differential equations is one of the most important fields of applications for integral equations. About a century ago the systematic development of the theory of integral equations was initiated by the treatment of boundary value problems and there has been an ongoing fruitful interaction between these two areas of applied mathematics. It is the aim of this chapter to introduce the main ideas of this field by studying the basic boundary value problems of potential theory. For the sake of simplicity we shall confine our presentation to the case of two and three space dimensions. The extension to more than three dimensions is straightforward. As we shall see, the treatment of the boundary integral equations for the potential theoretic boundary value problems delivers an instructive example for the application of the Fredholm alternative, since both its cases occur in a natural way.

6.1 Harmonic Functions

We begin with a brief outline of the basic properties of harmonic functions going back to the early development of potential theory at the beginning of the 19th century with contributions by Dirichlet, Gauss, Green, Riemann and Weierstrass. For a more comprehensive study of potential theory we refer to Courant and Hilbert [27], Folland [41], Helms [66], Kellogg [82], Martensen [118], and Mikhlin [123].

Definition 6.1 A twice continuously differentiable real-valued function u , defined on a domain $D \subset \mathbb{R}^m$, $m = 2, 3$, is called harmonic if it satisfies Laplace's equation

$$\Delta u = 0 \quad \text{in } D,$$

where

$$\Delta u := \sum_{j=1}^m \frac{\partial^2 u}{\partial x_j^2}.$$

Harmonic functions describe time-independent temperature distributions, potentials of electrostatic and magnetostatic fields, and velocity potentials of incompressible irrotational fluid flows.

There is a close connection between harmonic functions in \mathbb{R}^2 and holomorphic functions in \mathbb{C} . From the Cauchy–Riemann equations we readily observe that both the real and imaginary parts of a holomorphic function $f(z) = u(x_1, x_2) + iv(x_1, x_2)$, $z = x_1 + ix_2$, are harmonic functions.

Most of the basic properties of harmonic functions can be deduced from the fundamental solution that is introduced in the following theorem. Recall that by $|x|$ we denote the Euclidean norm of a vector $x \in \mathbb{R}^m$.

Theorem 6.2 *The function*

$$\Phi(x, y) := \begin{cases} \frac{1}{2\pi} \ln \frac{1}{|x - y|}, & m = 2, \\ \frac{1}{4\pi} \frac{1}{|x - y|}, & m = 3, \end{cases}$$

is called the fundamental solution of Laplace's equation. For fixed $y \in \mathbb{R}^m$ it is harmonic in $\mathbb{R}^m \setminus \{y\}$.

Proof. This follows by straightforward differentiation. \square

For $n \in \mathbb{N}$, by $C^n(D)$ we denote the linear space of real- or complex-valued functions defined on the domain D , which are n times continuously differentiable. By $C^n(\bar{D})$ we denote the subspace of all functions in $C^n(D)$, which with all their derivatives up to order n can be extended continuously from D into the closure \bar{D} . In this chapter, we mostly deal with real-valued functions but with proper interpretation our results remain valid for complex-valued functions. From p. 25 we recall what is meant by saying a bounded domain D or its boundary ∂D belong to class C^n for $n \in \mathbb{N}$.

One of the basic tools in studying harmonic functions is provided by Green's integral theorems. Recall that for two vectors $a = (a_1, \dots, a_m)$ and $b = (b_1, \dots, b_m)$ in \mathbb{R}^m we denote by $a \cdot b = a_1 b_1 + \dots + a_m b_m$ the dot product.

Theorem 6.3 (Green's Theorem) Let D be a bounded domain of class C^1 and let ν denote the unit normal vector to the boundary ∂D directed into the exterior of D . Then, for $u \in C^1(\bar{D})$ and $v \in C^2(\bar{D})$ we have Green's first theorem

$$\int_D \{u \Delta v + \operatorname{grad} u \cdot \operatorname{grad} v\} dx = \int_{\partial D} u \frac{\partial v}{\partial \nu} ds \quad (6.1)$$

and for $u, v \in C^2(\bar{D})$ we have Green's second theorem

$$\int_D (u \Delta v - v \Delta u) dx = \int_{\partial D} \left(u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \right) ds. \quad (6.2)$$

Proof. We apply Gauss' divergence theorem

$$\int_D \operatorname{div} A dx = \int_{\partial D} \nu \cdot A ds$$

to the vector field $A \in C^1(\bar{D})$ defined by $A := u \operatorname{grad} v$ and use

$$\operatorname{div}(u \operatorname{grad} v) = \operatorname{grad} u \cdot \operatorname{grad} v + u \operatorname{div} \operatorname{grad} v$$

to establish (6.1). To obtain (6.2) we interchange u and v and then subtract. \square

Note that our regularity assumptions on D are sufficient conditions for the validity of Gauss' and Green's theorems and can be weakened. In particular, the boundary can be allowed to have edges and corners. For a detailed study, see, for example, [123, 130].

Corollary 6.4 Let $v \in C^2(\bar{D})$ be harmonic in D . Then

$$\int_{\partial D} \frac{\partial v}{\partial \nu} ds = 0. \quad (6.3)$$

Proof. This follows by choosing $u = 1$ in (6.1). \square

Theorem 6.5 (Green's Formula) Let D be as in Theorem 6.3 and let $u \in C^2(\bar{D})$ be harmonic in D . Then

$$u(x) = \int_{\partial D} \left\{ \frac{\partial u}{\partial \nu}(y) \Phi(x, y) - u(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)} \right\} ds(y), \quad x \in D. \quad (6.4)$$

Proof. For $x \in D$ we choose a sphere $\Omega(x; r) := \{y \in \mathbb{R}^m : |y - x| = r\}$ of radius r such that $\Omega(x; r) \subset D$ and direct the unit normal ν to $\Omega(x; r)$ into the interior of $\Omega(x; r)$. Now we apply Green's second theorem (6.2) to the harmonic functions u and $\Phi(x, \cdot)$ in the domain $\{y \in D : |y - x| > r\}$ to obtain

$$\int_{\partial D \cup \Omega(x; r)} \left\{ u(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)} - \frac{\partial u}{\partial \nu}(y) \Phi(x, y) \right\} ds(y) = 0.$$

Since on $\Omega(x; r)$ we have

$$\operatorname{grad}_y \Phi(x, y) = \frac{\nu(y)}{\omega_m r^{m-1}}, \quad (6.5)$$

where $\omega_2 = 2\pi$, $\omega_3 = 4\pi$, a straightforward calculation, using the mean value theorem and (6.3), shows that

$$\lim_{r \rightarrow 0} \int_{\Omega(x;r)} \left\{ u(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)} - \frac{\partial u}{\partial \nu}(y) \Phi(x, y) \right\} ds(y) = u(x),$$

whence (6.4) follows. \square

From Green's formula we can conclude that harmonic functions are analytic functions of their independent variables.

Theorem 6.6 *Harmonic functions are analytic.*

Proof. We make use of the fact that each holomorphic function of several complex variables, i.e., a function satisfying the Cauchy–Riemann equations with respect to each of the complex variables is also analytic, i.e., it has a local power series expansion and vice versa (see [50]). The theorem now follows from the observation that the fundamental solution $\Phi(x, y)$ is an analytic function of the Cartesian coordinates x_j , $j = 1, \dots, m$, of x and the fact that the integrands in (6.4) and their derivatives with respect to x are continuous with respect to y if x is contained in a compact subset of D . Therefore the Cauchy–Riemann equations for u can be verified by differentiating with respect to x under the integral. \square

From Theorem 6.6 it follows that a harmonic function that vanishes in an open subset of its domain of definition must vanish identically.

Theorem 6.7 (Mean Value Theorem) *Let u be harmonic in an open ball $B(x; r) = \{y \in \mathbb{R}^m : |y - x| < r\}$ with boundary $\Omega(x; r)$ and continuous in the closure $B[x; r]$. Then*

$$u(x) = \frac{m}{\omega_m r^m} \int_{B[x;r]} u(y) dy = \frac{1}{\omega_m r^{m-1}} \int_{\Omega(x;r)} u(y) ds(y), \quad (6.6)$$

i.e., the value of u at the center of the ball is equal to the integral mean values over both the ball and its boundary surface ($\omega_2 = 2\pi$, $\omega_3 = 4\pi$).

Proof. For each $0 < \rho < r$ we have $u \in C^2(B[x; \rho])$ and can apply (6.3) and (6.4) with the result

$$u(x) = \frac{1}{\omega_m \rho^{m-1}} \int_{|y-x|= \rho} u(y) ds(y), \quad (6.7)$$

whence the second mean value formula follows by passing to the limit $\rho \rightarrow r$. Multiplying (6.7) by ρ^{m-1} and integrating with respect to ρ from 0 to r we obtain the first mean value formula. \square

Theorem 6.8 (Maximum-Minimum Principle) *A harmonic function on a domain cannot attain its maximum or its minimum unless it is constant.*

Proof. It suffices to carry out the proof for the maximum. Let u be a harmonic function in the domain D and assume that it attains its maximum value in D , i.e., the set $D_M := \{x \in D : u(x) = M\}$ where $M := \sup_{x \in D} u(x)$ is not empty. Since u is continuous, D_M is closed relative to D . Let x be any point in D_M and apply the mean value Theorem 6.7 to the harmonic function $M - u$ in a ball $B(x; r)$ with $B[x; r] \subset D$. Then

$$0 = M - u(x) = \frac{m}{\omega_m r^m} \int_{B[x;r]} \{M - u(y)\} dy,$$

so that $u = M$ in $B(x; r)$. Therefore D_M is open relative to D . Hence $D = D_M$, i.e., u is constant in D . \square

Corollary 6.9 *Let D be a bounded domain and let u be harmonic in D and continuous in \bar{D} . Then u attains both its maximum and its minimum on the boundary.*

For the study of exterior boundary value problems we also need to investigate the asymptotic behavior of harmonic functions as $|x| \rightarrow \infty$. To this end we extend Green's formula to unbounded domains.

Theorem 6.10 *Assume that D is a bounded domain of class C^1 with a connected boundary ∂D and outward unit normal ν and let $u \in C^2(\mathbb{R}^m \setminus D)$ be a bounded harmonic function. Then*

$$u(x) = u_\infty + \int_{\partial D} \left\{ u(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)} - \frac{\partial u}{\partial \nu}(y) \Phi(x, y) \right\} ds(y) \quad (6.8)$$

for $x \in \mathbb{R}^m \setminus D$ and some constant u_∞ . For $m = 2$, in addition,

$$\int_{\partial D} \frac{\partial u}{\partial \nu} ds = 0$$

and the mean value property at infinity

$$u_\infty = \frac{1}{2\pi r} \int_{|y|=r} u(y) ds(y)$$

for sufficiently large r is satisfied.

Proof. Without loss of generality we may assume that the origin $x = 0$ is contained in D . Since u is bounded, there exists a constant $M > 0$ such that $|u(x)| \leq M$ for all $x \in \mathbb{R}^m \setminus D$. Choose R_0 large enough to ensure that $y \in \mathbb{R}^m \setminus D$ for all $|y| \geq R_0/2$. Then for a fixed x with $|x| \geq R_0$ we can apply the mean value Theorem 6.7 to the components of $\text{grad } u$. From this and Gauss' integral theorem we obtain

$$\text{grad } u(x) = \frac{m}{\omega_m r^m} \int_{B(x;r)} \text{grad } u(y) dy = -\frac{m}{\omega_m r^m} \int_{\Omega(x;r)} \nu(y) u(y) ds(y),$$

where ν is the unit normal to $\Omega(x;r)$ directed into the interior of $\Omega(x;r)$ and where we choose the radius to be $r = |x|/2$. Then we can estimate

$$|\text{grad } u(x)| \leq \frac{mM}{r} = \frac{2mM}{|x|} \quad (6.11)$$

for all $|x| \geq R_0$.

For $m = 2$, we choose r large enough such that $\Omega_r := \Omega(0;r)$ is contained in $\mathbb{R}^2 \setminus \bar{D}$ and apply Green's second theorem (6.2) to u and $\Phi(0,\cdot)$ in the annulus $r < |y| < R$ and use (6.5) to obtain

$$\frac{1}{r} \int_{\Omega_r} u ds - \ln \frac{1}{r} \int_{\Omega_r} \frac{\partial u}{\partial \nu} ds = \frac{1}{R} \int_{\Omega_R} u ds - \ln \frac{1}{R} \int_{\Omega_R} \frac{\partial u}{\partial \nu} ds.$$

(Note that ν is the interior normal to Ω_r and Ω_R .) From this, with the aid of Corollary 6.4 applied in the annulus between ∂D and Ω_r , we find

$$\frac{1}{r} \int_{\Omega_r} u ds + \ln \frac{1}{r} \int_{\partial D} \frac{\partial u}{\partial \nu} ds = \frac{1}{R} \int_{\Omega_R} u ds + \ln \frac{1}{R} \int_{\partial D} \frac{\partial u}{\partial \nu} ds. \quad (6.12)$$

Since the first term on the right-hand side is bounded by $2\pi M$, letting $R \rightarrow \infty$ in (6.12) implies that the integral in (6.9) must be zero. Note that (6.9) only holds in the two-dimensional case and is a consequence of the fact that in \mathbb{R}^2 the fundamental solution is not bounded at infinity.

For $x \in \mathbb{R}^m \setminus \bar{D}$, $m = 2, 3$, we now choose r large enough such that $\bar{D} \subset B(x;r)$. Then by Green's formula (6.4), applied in the domain between ∂D and $\Omega(x;r)$, we have that

$$u(x) = \int_{\partial D \cup \Omega(x;r)} \left\{ u(y) \frac{\partial \Phi(x,y)}{\partial \nu(y)} - \frac{\partial u}{\partial \nu}(y) \Phi(x,y) \right\} ds(y). \quad (6.13)$$

With the aid of Corollary 6.4 we find

$$\int_{\Omega(x;r)} \frac{\partial u}{\partial \nu}(y) \Phi(x,y) ds(y) = \frac{1}{4\pi r} \int_{\partial D} \frac{\partial u}{\partial \nu}(y) ds(y) \rightarrow 0, \quad r \rightarrow \infty,$$

if $m = 3$; and

$$\int_{\Omega(x;r)} \frac{\partial u}{\partial \nu}(y) \Phi(x,y) ds(y) = \frac{1}{2\pi} \ln \frac{1}{r} \int_{\partial D} \frac{\partial u}{\partial \nu}(y) ds(y) = 0$$

if $m = 2$, where we have made use of (6.9). With the aid of (6.5) we can write

$$\int_{\Omega(x;r)} u(y) \frac{\partial \Phi(x,y)}{\partial \nu(y)} ds(y) = \frac{1}{\omega_m r^{m-1}} \int_{|y-x|=r} u(y) ds(y).$$

From the mean value theorem we have

$$u(x+y) - u(y) = \text{grad } u(y+\theta x) \cdot x$$

for some $\theta \in [0, 1]$, and using (6.11) we can estimate

$$|u(x+y) - u(y)| \leq \frac{2mM|x|}{|y|-|x|}$$

provided that $|y|$ is sufficiently large. Therefore

$$\frac{1}{\omega_m r^{m-1}} \left| \int_{|y-x|=r} u(y) ds(y) - \int_{|y|=r} u(y) ds(y) \right| \leq \frac{C}{r}$$

for some constant $C > 0$ depending on x and all sufficiently large r . Now choose a sequence (r_n) of radii with $r_n \rightarrow \infty$. Since the integral mean values

$$\mu_n := \frac{1}{\omega_m r_n^{m-1}} \int_{|y|=r_n} u(y) ds(y)$$

are bounded through $|\mu_n| \leq M$, $n \in \mathbb{N}$, by the Bolzano–Weierstrass theorem we may assume that the sequence (μ_n) converges, i.e., $\mu_n \rightarrow u_\infty$, $n \rightarrow \infty$, for some $u_\infty \in \mathbb{R}$. From this, in view of the above estimates, we now have that

$$\int_{\Omega(x;r_n)} \left\{ u(y) \frac{\partial \Phi(x,y)}{\partial \nu(y)} - \frac{\partial u}{\partial \nu}(y) \Phi(x,y) \right\} ds(y) \rightarrow u_\infty, \quad n \rightarrow \infty.$$

Hence (6.8) follows by setting $r = r_n$ in (6.13) and passing to the limit $n \rightarrow \infty$. Finally, (6.10) follows by setting $R = r_n$ in (6.12), passing to the limit $n \rightarrow \infty$, and using (6.9). \square

From (6.8), using the asymptotic behavior of the fundamental solution

$$\Phi(x,y) := \begin{cases} \frac{1}{2\pi} \ln \frac{1}{|x|} + O\left(\frac{1}{|x|}\right), & m = 2, \\ O\left(\frac{1}{|x|}\right), & m = 3, \end{cases} \quad (6.14)$$

$$\text{and } \frac{\partial \Phi(x,y)}{\partial x_j} = O\left(\frac{1}{|x|^{m-1}}\right), \quad \frac{\partial^2 \Phi(x,y)}{\partial x_j \partial x_k} = O\left(\frac{1}{|x|^m}\right) \quad (6.15)$$

for $|x| \rightarrow \infty$, which holds uniformly for all directions $x/|x|$ and all $y \in \partial D$, and the property (6.9) if $m = 2$, we can deduce that bounded harmonic functions in an exterior domain satisfy

$$u(x) = u_\infty + O\left(\frac{1}{|x|}\right), \quad \text{grad } u(x) = O\left(\frac{1}{|x|^{m-1}}\right), \quad |x| \rightarrow \infty, \quad (6.16)$$

uniformly for all directions.

6.2 Boundary Value Problems: Uniqueness

Green's formula (6.4) represents any harmonic function in terms of its boundary values and its normal derivative on the boundary, the so-called *Cauchy data*. In the subsequent analysis we shall see that a harmonic function is already completely determined by either its boundary values or, up to a constant, its normal derivative alone.

In the sequel, let $D \subset \mathbb{R}^m$ be a bounded domain of class C^2 . For the sake of simplicity for the rest of this chapter we assume that the boundary ∂D is connected. Again by ν we denote the unit normal of ∂D directed into the exterior domain $\mathbb{R}^m \setminus \bar{D}$.

Interior Dirichlet Problem. Find a function u that is harmonic in D , is continuous in \bar{D} , and satisfies the boundary condition

$$u = f \quad \text{on } \partial D,$$

where f is a given continuous function.

Interior Neumann Problem. Find a function u that is harmonic in D , is continuous in \bar{D} , and satisfies the boundary condition

$$\frac{\partial u}{\partial \nu} = g \quad \text{on } \partial D$$

in the sense

$$\lim_{h \rightarrow +0} \nu(x) \cdot \text{grad } u(x - h\nu(x)) = g(x), \quad x \in \partial D,$$

of uniform convergence on ∂D , where g is a given continuous function.

Exterior Dirichlet Problem. Find a function u that is harmonic in $\mathbb{R}^m \setminus \bar{D}$, is continuous in $\mathbb{R}^m \setminus D$, and satisfies the boundary condition

$$u = f \quad \text{on } \partial D,$$

for $|x| \rightarrow \infty$, which holds uniformly for all directions $x/|x|$ and all $y \in \partial D$, and the property (6.9) if $m = 2$, we can deduce that bounded harmonic functions in an exterior domain satisfy

$$u(x) = O(1), \quad m = 2, \quad \text{and} \quad u(x) = o(1), \quad m = 3,$$

uniformly for all directions.

Exterior Neumann Problem. Find a function u that is harmonic in $\mathbb{R}^m \setminus \bar{D}$, is continuous in $\mathbb{R}^m \setminus D$, and satisfies the boundary condition

$$\frac{\partial u}{\partial \nu} = g \quad \text{on } \partial D$$

in the sense of uniform convergence on ∂D , where g is a given continuous function. For $|x| \rightarrow \infty$ it is required that $u(x) = o(1)$ uniformly for all directions.

Note that for the exterior problems we impose that $u_\infty = 0$, with the exception of the Dirichlet problem in \mathbb{R}^2 , where u is only required to be bounded.

These boundary value problems carry the names of Dirichlet, who made important contributions to potential theory, and Neumann, who gave the first rigorous existence proof (see Problem 6.5). From the numerous applications we mention:

- (1) Determine the stationary temperature distribution in a heat-conducting body from the temperature on the boundary or from the heat flux through the boundary.
- (2) Find the potential of the electrostatic field in the exterior of a perfect conductor.
- (3) Find the velocity potential of an incompressible irrotational flow around an obstacle.

Our aim is to establish that each of the above potential theoretic boundary value problems has a unique solution depending continuously on the given boundary data, i.e., they are well-posed in the sense of Hadamard (see Section 15.1).

In our uniqueness proofs we need to apply Green's Theorem 6.3. Since for solutions to the boundary value problems we do not assume differentiability up to the boundary, we introduce the concept of *parallel surfaces*. These are described by

$$\partial D_h := \{z = x + hv(x) : x \in \partial D\},$$

with a real parameter h . Because ∂D is assumed to be of class C^2 , we observe that ∂D_h is of class C^1 . For $m = 3$, let $\mathbf{x}(u) = (x_1(u), x_2(u), x_3(u))$, $u = (u_1, u_2)$, be a regular parametric representation of a surface patch of ∂D . Then straightforward differential geometric calculations show that the determinants

$$g(u) := \det \left[\frac{\partial \mathbf{x}}{\partial u_i} \cdot \frac{\partial \mathbf{x}}{\partial u_j} \right] \quad \text{and} \quad g(u; h) := \det \left[\frac{\partial \mathbf{z}}{\partial u_i} \cdot \frac{\partial \mathbf{z}}{\partial u_j} \right]$$

are related by

$$g(u, h) = g(u)\{1 - 2hH(u) + h^2K(u)\}^2,$$

where H and K denote the mean and Gaussian curvature of ∂D , respectively (see [118, 130]). This verifies that the parallel surfaces are well defined provided the parameter h is sufficiently small to ensure that $1 - 2hH + h^2K$ remains positive. This also ensures that in a sufficiently small neighborhood of ∂D each point z can be uniquely represented in the form $z = x + h\nu(x)$, where $x \in \partial D$ and $h \in \mathbb{R}$.

In particular, the surface elements ds on ∂D and ds_h on ∂D_h are related by

$$(6.17) \quad ds_h(z) = \{1 - 2hH + h^2K\}ds(x).$$

Since $\nu(x) \cdot \nu(x) = 1$, we have

$$\frac{\partial \nu(x)}{\partial u_i} \cdot \nu(x) = 0, \quad i = 1, 2,$$

for all $x \in \partial D$, and therefore the tangential vectors

$$\frac{\partial z}{\partial u_i} = \frac{\partial x}{\partial u_i} + h \frac{\partial \nu(x)}{\partial u_i}, \quad i = 1, 2,$$

for all (sufficiently small) h lie in the tangent plane to ∂D at the point x , i.e., the normal vector $\nu_h(z)$ of the parallel surface ∂D_h coincides with the normal vector $\nu(x)$ of ∂D for all $x \in \partial D$. Hence, in view of (6.17), Theorems 6.5 and 6.10 remain valid for harmonic functions $u \in C(\bar{D})$ and $u \in C(\mathbb{R}^m \setminus D)$, respectively, provided they have a normal derivative in the sense of uniform convergence.

Note that in two dimensions the equation (6.17) has to be replaced by $ds_h(z) = (1 - \kappa h)ds(x)$, where κ denotes the curvature of ∂D , i.e., for the representation $\partial D = \{x(s) : s_0 \leq s \leq s_1\}$ in terms of the arc length we have $\kappa = \nu \cdot \nu''$.

Theorem 6.11 Both the interior and the exterior Dirichlet problems have at most one solution.

Proof. The difference $u := u_1 - u_2$ of two solutions to the Dirichlet problem is a harmonic function that is continuous up to the boundary and satisfies the homogeneous boundary condition $u = 0$ on ∂D . Then, from the maximum-minimum principle of Corollary 6.9 we obtain $u = 0$ in D for the interior problem, and observing that $u(x) = o(1)$, $|x| \rightarrow \infty$, we also obtain $u = 0$ in $\mathbb{R}^3 \setminus D$ for the exterior problem in three dimensions.

For the exterior problem in two dimensions, by the maximum-minimum principle Theorem 6.8 the supremum and the infimum of the bounded harmonic function u are either attained on the boundary or equal to u_{∞} . When the maximum and minimum are both attained on the boundary then

from the homogeneous boundary condition we immediately have $u = 0$ in $\mathbb{R}^m \setminus \bar{D}$. If the supremum is equal to u_{∞} , then from $u(x) \leq u_{\infty}$ for all $x \in \mathbb{R}^2 \setminus D$ and the mean value property (6.10) we observe that $u = u_{\infty}$ in the exterior of some circle. Now we can apply the maximum principle to see that $u = u_{\infty}$ in all of $\mathbb{R}^2 \setminus D$ and the homogeneous boundary condition finally implies $u = 0$ in $\mathbb{R}^2 \setminus D$. The case where the infimum is equal to u_{∞} is settled by the same argument. \square

Theorem 6.12 Two solutions of the interior Neumann problem can differ only by a constant. The exterior Neumann problem has at most one solution.

Proof. The difference $u := u_1 - u_2$ of two solutions for the Neumann problem is a harmonic function continuous up to the boundary satisfying the homogeneous boundary condition $\partial u / \partial \nu = 0$ on ∂D in the sense of uniform convergence. For the interior problem, suppose that u is not constant in D . Then there exists some closed ball B contained in D such that $\int_B |\operatorname{grad} u|^2 dx > 0$. From Green's first theorem (6.1), applied to the interior D_h of some parallel surface $\partial D_h := \{x - h\nu(x) : x \in \partial D\}$ with sufficiently small $h > 0$, we derive

$$\int_B |\operatorname{grad} u|^2 dx \leq \int_{D_h} |\operatorname{grad} u|^2 dx = \int_{\partial D_h} u \frac{\partial u}{\partial \nu} ds.$$

Passing to the limit $h \rightarrow 0$, we obtain the contradiction $\int_B |\operatorname{grad} u|^2 dx \leq 0$. Hence, u must be constant.

For the exterior problem, assume that $\operatorname{grad} u \neq 0$ in $\mathbb{R}^m \setminus \bar{D}$. Then again, there exists some closed ball B contained in $\mathbb{R}^m \setminus \bar{D}$ such that $\int_B |\operatorname{grad} u|^2 dx > 0$. From Green's first theorem, applied to the domain $D_{h,r}$ between some parallel surface $\partial D_h := \{x + h\nu(x) : x \in \partial D\}$ with sufficiently small $h > 0$ and some sufficiently large sphere Ω_r of radius r centered at the origin (with interior normal ν), we obtain

$$\int_B |\operatorname{grad} u|^2 dx \leq \int_{D_{h,r}} |\operatorname{grad} u|^2 dx = - \int_{\Omega_r} u \frac{\partial u}{\partial \nu} ds - \int_{\partial D_h} u \frac{\partial u}{\partial \nu} ds.$$

Letting $r \rightarrow \infty$ and $h \rightarrow 0$, with the aid of the asymptotics (6.16), we arrive at the contradiction $\int_B |\operatorname{grad} u|^2 dx \leq 0$. Therefore, u is constant in $\mathbb{R}^m \setminus \bar{D}$ and the constant must be zero, since $u_{\infty} = 0$. \square

From the proofs it is obvious that our uniqueness results remain valid under weaker regularity conditions on the boundary. Uniqueness for the Dirichlet problem via the maximum-minimum principle needs no regularity of the boundary, and uniqueness for the Neumann problem holds for those boundaries for which Green's integral theorem is valid. We have formulated the boundary value problems for C^2 boundaries, since we shall establish the existence of solutions under these conditions.

6.3 Surface Potentials

Definition 6.13 Given a function $\varphi \in C(\partial D)$, the functions

$$u(x) := \int_{\partial D} \varphi(y) \Phi(x, y) ds(y), \quad x \in \mathbb{R}^m \setminus \partial D, \quad (6.18)$$

and

$$v(x) := \int_{\partial D} \varphi(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)} ds(y), \quad x \in \mathbb{R}^m \setminus \partial D, \quad (6.19)$$

are called, respectively, single-layer and double-layer potential with density φ . In two dimensions, occasionally, for obvious reasons we will call them logarithmic single-layer and logarithmic double-layer potential.

For fixed $y \in \mathbb{R}^m$ the fundamental solution $u = \Phi(\cdot, y)$ represents the potential of a unit point source located at the point y , i.e., $\text{grad}_x \Phi(x, y)$ gives the force-field of this point source acting at the point x . The single-layer potential is obtained by distributing point sources on the boundary ∂D . For $h > 0$, by the mean value theorem we have

$$\Phi(x, y + h\nu(y)) - \Phi(x, y - h\nu(y)) = 2h \nu(y) \cdot \text{grad } \Phi(x, y + \theta h\nu(y))$$

for some $\theta = \theta(y) \in [-1, 1]$. Therefore, the double-layer potential can be interpreted as the limit $h \rightarrow 0$ of the superposition of the single-layer potentials u_h and u_{-h} with densities $\varphi/2h$ on ∂D_h and $-\varphi/2h$ on ∂D_{-h} , respectively, i.e., the double-layer potential is obtained by distributing dipoles on the boundary ∂D .

Since for points $x \notin \partial D$ we can interchange differentiation and integration, the single- and double-layer potentials represent harmonic functions in D and $\mathbb{R}^m \setminus \bar{D}$. For the solution of the boundary value problems we need to investigate the behavior of the potentials at the boundary ∂D where the integrals become singular. The boundary behavior is expressed by the following so-called jump relations.

Theorem 6.14 Let ∂D be of class C^2 and $\varphi \in C(\partial D)$. Then the single-layer potential u with density φ is continuous throughout \mathbb{R}^m . On the boundary we have

$$u(x) = \int_{\partial D} \varphi(y) \Phi(x, y) ds(y), \quad x \in \partial D, \quad (6.20)$$

where the integral exists as an improper integral.

Proof. Analogous to the proofs of Theorems 2.22 and 2.23, by using the cut-off function h , it can be shown that the single-layer potential u is the uniform limit of a sequence of functions u_n that are continuous in \mathbb{R}^m . \square

For the further analysis of the jump relations we need the following lemma. The inequality (6.21) expresses the fact that the vector $x - y$ for x close to y is almost orthogonal to the normal vector $\nu(x)$.

Lemma 6.15 Let ∂D be of class C^2 . Then there exists a positive constant L such that

$$|\nu(x) \cdot \{x - y\}| \leq L|x - y|^2 \quad (6.21)$$

and

$$|\nu(x) - \nu(y)| \leq L|x - y| \quad (6.22)$$

for all $x, y \in \partial D$.

Proof. We confine ourselves to the two-dimensional case. For the three-dimensional case we refer to [24]. Let $\Gamma = \{x(s) : s \in [0, s_0]\}$ be a regular parameterization of a patch $\Gamma \subset \partial D$, i.e., $x : [0, 1] \rightarrow \Gamma \subset \partial D$ is injective and twice continuously differentiable with $x'(s) \neq 0$ for all $s \in [0, s_0]$. Then, by Taylor's formula we have

$$|\nu(x(t)) \cdot \{x(t) - x(\tau)\}| \leq \frac{1}{2} \max_{0 \leq s \leq s_0} |x''(s)| |t - \tau|^2, \quad \text{exists by Taylor's formula}$$

$$|\nu(x(t)) - \nu(x(\tau))| \leq \max_{0 \leq s \leq s_0} \left| \frac{d}{ds} \nu(x(s)) \right| |t - \tau|, \quad \text{exists by Taylor's formula}$$

$$|\nu(x(t)) - x(\tau)| \geq \min_{0 \leq s \leq s_0} |x'(s)| |t - \tau|. \quad \text{needed to get}$$

The statement of the lemma is evident from this. \square

Example 6.16 For the double-layer potential with constant density we have

$$2 \int_{\partial D} \frac{\partial \Phi(x, y)}{\partial \nu(y)} ds(y) = \begin{cases} -2, & x \in D, \\ -1, & x \in \partial D, \end{cases} \quad (6.23)$$

This follows for $x \in \mathbb{R}^m \setminus \bar{D}$ from (6.3) applied to $\Phi(x, \cdot)$ and for $x \in D$ from (6.4) applied to $u = 1$ in D . The result for $x \in \partial D$ is derived by excluding x from the integration by circumscribing it with a sphere $\Omega(x; r)$ of radius r and center x with the unit normal directed toward the center. \square

Let $H(x; r) := \Omega(x; r) \cap D$. Then, by (6.3) applied to $\Phi(x, \cdot)$, we have

$$\int_{\{y \in \partial D : |y - x| \geq r\}} \frac{\partial \Phi(x, y)}{\partial \nu(y)} ds(y) + \int_{H(x; r)} \frac{\partial \Phi(x, y)}{\partial \nu(y)} ds(y) = 0, \quad \text{since } \Phi(x, \cdot) \text{ is harmonic inside } D,$$

and from

$$\lim_{r \rightarrow 0} 2 \int_{H(x; r)} \frac{\partial \Phi(x, y)}{\partial \nu(y)} ds(y) = \lim_{r \rightarrow 0} \frac{2}{\omega_m r^{m-1}} \int_{H(x; r)} ds(y) = 1 \quad \text{from where?}$$

Theorem 6.17 For ∂D of class C^2 , the double-layer potential v with continuous density φ can be continuously extended from D to \bar{D} and from $\mathbb{R}^m \setminus \bar{D}$ to $\mathbb{R}^m \setminus D$ with limiting values

$$v_{\pm}(x) = \int_{\partial D} \varphi(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)} ds(y) \pm \frac{1}{2} \varphi(x), \quad x \in \partial D, \quad (6.24)$$

where

$$v_{\pm}(x) := \lim_{h \rightarrow 0} v(x \pm h\nu(x))$$

and where the integral exists as an improper integral.

Proof. Because of Lemma 6.15 we have the estimate

$$\left| \frac{\partial \Phi(x, y)}{\partial \nu(y)} \right| = \frac{|\nu(y) \cdot \{x - y\}|}{\omega_m |x - y|^m} \leq \frac{L}{\omega_m |x - y|^{m-2}}, \quad x \neq y, \quad (6.25)$$

i.e., the integral in (6.24) has a weakly singular kernel. Therefore, by Theorem 2.23 the integral exists for $x \in \partial D$ as an improper integral and represents a continuous function on ∂D .

As pointed out on p. 76, in a sufficiently small neighborhood U of ∂D we can represent each $x \in U$ uniquely in the form $x = z + h\nu(z)$, where $z \in \partial D$ and $h \in [-h_0, h_0]$ for some $h_0 > 0$. Then we write the double-layer potential v with density φ in the form

$$v(x) = \varphi(z)w(x) + u(x), \quad x = z + h\nu(z) \in U \setminus \partial D,$$

where

$$w(x) := \int_{\partial D} \frac{\partial \Phi(x, y)}{\partial \nu(y)} ds(y)$$

and

$$u(x) := \int_{\partial D} \{\varphi(y) - \varphi(z)\} \frac{\partial \Phi(x, y)}{\partial \nu(y)} ds(y). \quad (6.26)$$

For $x \in \partial D$, i.e., for $h = 0$, the integral in (6.26) exists as an improper integral and represents a continuous function on ∂D . Therefore, in view of Example 6.16, to establish the theorem it suffices to show that

$$\lim_{h \rightarrow 0} u(z + h\nu(z)) = u(z), \quad z \in \partial D,$$

uniformly on ∂D .

From (6.21) we can conclude that

$$|x - y|^2 \geq \frac{1}{2} \{|z - y|^2 + |x - z|^2\}$$

for $x = z + h\nu(z)$ and $h \in [-h_0, h_0]$ provided that h_0 is sufficiently small. Therefore, writing

$$\frac{\partial \Phi(x, y)}{\partial \nu(y)} = \frac{\nu(y) \cdot \{z - y\}}{\omega_m |x - y|^m} + \frac{\nu(y) \cdot \{x - z\}}{\omega_m |x - y|^m},$$

and again using (6.21), we can estimate

$$\left| \frac{\partial \Phi(x, y)}{\partial \nu(y)} \right| \leq C_1 \left\{ \frac{1}{|x - y|^{m-2}} + \frac{|x - z|}{\|z - y\|^2 + |x - z|^2 m/2} \right\}$$

for some constant $C_1 > 0$. Recalling the proof of Theorem 2.23 and denoting $\partial D(z; r) := \partial D \cap B(z; r)$, for sufficiently small r we project onto the tangent plane and deduce that

$$\int_{\partial D(x; r)} \left| \frac{\partial \Phi(x, y)}{\partial \nu(y)} \right| ds(y) \leq C_1 \left\{ \int_0^r d\rho + \int_0^r \frac{|x - z| \rho^{m-2} d\rho}{(\rho^2 + |x - z|^2)^{m/2}} \right\}$$

$$\leq C_1 \left\{ r + \int_0^\infty \frac{\lambda^{m-2} d\lambda}{(\lambda^2 + 1)^{m/2}} \right\}. \quad (6.27)$$

From the mean value theorem we obtain that

$$\text{Analog applied to } \Phi_{\nu}(x, y) \quad \left| \frac{\partial \Phi(x, y)}{\partial \nu(y)} - \frac{\partial \Phi(z, y)}{\partial \nu(y)} \right| \leq C_2 \frac{|x - z|}{|z - y|^m} \quad \begin{array}{l} \text{blows up as } r \rightarrow 0 \\ \text{as func of } h. \end{array}$$

for some constant $C_2 > 0$ and $2|x - z| \leq |z - y|$. Hence we can estimate

$$\int_{\partial D \setminus \partial D(x; r)} \left| \frac{\partial \Phi(x, y)}{\partial \nu(y)} - \frac{\partial \Phi(z, y)}{\partial \nu(y)} \right| ds(y) \leq C_3 \frac{|x - z|}{r^m} \quad (6.28)$$

for some constant $C_3 > 0$ and $|x - z| \leq r/2$. Now we can combine (6.27) and (6.28) to find that

$$|u(x) - u(z)| \leq C \left\{ \max_{|y-x| \leq r} |\varphi(y) - \varphi(z)| + \frac{|x - z|}{r^m} \right\}$$

for some constant $C > 0$, all sufficiently small r , and $|x - z| \leq r/2$. Given $\varepsilon > 0$ we can choose $r > 0$ such that

$$\max_{|y-x| \leq r} |\varphi(y) - \varphi(z)| \leq \frac{\varepsilon}{2C} \quad \text{if } \varphi \in \mathcal{C}(\mathbb{R}).$$

for all $z \in \partial D$, since φ is uniformly continuous on ∂D . Then, taking $\delta < \varepsilon r^m / 2C$, we see that $|u(x) - u(z)| < \varepsilon$ for all $|x - z| < \delta$, and the proof is complete. \square

Theorem 6.18 Let ∂D be of class C^2 . Then for the single-layer potential u with continuous density φ we have

$$\frac{\partial u_{\pm}}{\partial \nu}(x) = \int_{\partial D} \varphi(y) \frac{\partial \Phi(x, y)}{\partial \nu(x)} ds(y) \mp \frac{1}{2} \varphi(x), \quad x \in \partial D, \quad (6.29)$$

where

$$\frac{\partial u_{\pm}}{\partial \nu}(x) := \lim_{h \rightarrow \pm 0} \nu(x) \cdot \text{grad } u(x \pm h\nu(x))$$

is to be understood in the sense of uniform convergence on ∂D and where the integral exists as an improper integral.

Proof. Let v denote the double-layer potential with density φ and let U be as in the proof of Theorem 6.17. Then for $x = z + h\nu(z) \in U \setminus \partial D$ we can write

$$\nu(z) \cdot \operatorname{grad} u(x) + v(x) = \int_{\partial D} \{\nu(y) - \nu(z)\} \cdot \operatorname{grad}_y \Phi(x, y) \varphi(y) ds(y),$$

where we have made use of $\operatorname{grad}_x \Phi(x, y) = -\operatorname{grad}_y \Phi(x, y)$. Using (6.22), analogous to the single-layer potential in Theorem 6.14, the right-hand side can be seen to be continuous in U . The proof is now completed by applying Theorem 6.17. \square

Theorem 6.19 *Let ∂D be of class C^2 . Then the double-layer potential v with continuous density φ satisfies*

$$\lim_{h \rightarrow +0} \nu(x) \cdot \{\operatorname{grad} v(x + h\nu(x)) - \operatorname{grad} v(x - h\nu(x))\} = 0 \quad (6.30)$$

uniformly for all $x \in \partial D$.

Proof. We omit the rather lengthy proof, which is similar in structure to the proof of Theorem 6.17. For a detailed proof we refer to [24]. \square

Only in $m=2$.

6.4 Boundary Value Problems: Existence

Green's formula shows that each harmonic function can be represented as a combination of single- and double-layer potentials. For boundary value problems we try to find a solution in the form of one of these two potentials. To this end we introduce two integral operators $K, K' : C(\partial D) \rightarrow C(\partial D)$ by

$$(K\varphi)(x) := 2 \int_{\partial D} \varphi(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)} ds(y), \quad x \in \partial D, \quad (6.31)$$

and

$$(K'\psi)(x) := 2 \int_{\partial D} \psi(y) \frac{\partial \Phi(x, y)}{\partial \nu(x)} ds(y), \quad x \in \partial D. \quad (6.32)$$

Because of (6.25) the integral operators K and K' have weakly singular kernels and therefore are compact by Theorem 2.23. Note that in two dimensions for C^2 boundaries the kernels of K and K' actually turn out to be continuous (see Problem 6.1). As seen by interchanging the order of integration, K and K' are adjoint with respect to the dual system $(C(\partial D), C(\partial D))$ defined by

$$\langle \varphi, \psi \rangle := \int_{\partial D} \varphi \psi ds, \quad \varphi, \psi \in C(\partial D).$$

Theorem 6.20 *The operators $I - K$ and $I - K'$ have trivial nullspaces*

$$N(I - K) = N(I - K') = \{0\}. \quad \text{i.e. 1 not eigen}$$

The nullspaces of the operators $I + K$ and $I + K'$ have dimension one and off K .

$$N(I + K) = \operatorname{span}\{1\}, \quad N(I + K') = \operatorname{span}\{\psi_0\}$$

with

$$\int_{\partial D} \psi_0 ds \neq 0,$$

i.e., the Riesz number is one.

Proof. Let φ be a solution to the homogeneous equation $\varphi - K\varphi = 0$ and define a double-layer potential v by (6.19). Then by (6.24) we have $2v_- = K\varphi - \varphi = 0$ and from the uniqueness for the interior Dirichlet problem (Theorem 6.11) it follows that $v = 0$ in D . From (6.30) we see that $\partial v_+ / \partial \nu = 0$ on ∂D , and since $v(x) = o(1)$, $|x| \rightarrow \infty$, from the uniqueness for the exterior Neumann problem (Theorem 6.12) we find that $v = 0$ in $\mathbb{R}^m \setminus \bar{D}$. Hence, from (6.24) we deduce $\varphi = v_+ - v_- = 0$ on ∂D . Thus $N(I - K) = \{0\}$ and, by the Fredholm alternative, $N(I - K') = \{0\}$.

Now let φ be a solution to $\varphi + K\varphi = 0$ and again define v by (6.19). Then by (6.24) we have $2v_+ = K\varphi + \varphi = 0$ on ∂D . Since $v(x) = o(1)$, $|x| \rightarrow \infty$, from the uniqueness for the exterior Dirichlet problem it follows that $v = 0$ in $\mathbb{R}^m \setminus \bar{D}$. From (6.30) we see that $\partial v_- / \partial \nu = 0$ on ∂D and from the uniqueness for the interior Neumann problem we find that v is constant in D . Hence, from (6.24) we deduce that φ is constant on ∂D . Therefore, $N(I + K) \subset \operatorname{span}\{1\}$, and since by (6.23) we have $1 + K1 = 0$, it follows that $N(I + K) = \operatorname{span}\{1\}$.

By the Fredholm alternative, $N(I + K')$ also has dimension one. Therefore $N(I + K') = \operatorname{span}\{\psi_0\}$ with some function $\psi_0 \in C(\partial D)$ that does not vanish identically. Assume that $\langle 1, \psi_0 \rangle = 0$ and define a single-layer potential u with density ψ_0 . Then by (6.20) and (6.29) we have

$$u_+ = u_-, \quad \frac{\partial u_-}{\partial \nu} = 0, \quad \text{and} \quad \frac{\partial u_+}{\partial \nu} = -\psi_0 \quad \text{on } \partial D \quad (6.33)$$

in the sense of uniform convergence. From $\partial u_- / \partial \nu = 0$ on ∂D , by the uniqueness for the interior Neumann problem (Theorem 6.12), we conclude that u is constant in D . Assume that u is not constant in $\mathbb{R}^m \setminus \bar{D}$. Then there exists a closed ball B contained in $\mathbb{R}^m \setminus \bar{D}$ such that $\int_B |\operatorname{grad} u|^2 dx > 0$. By Green's theorem (6.1), using the jump relations (6.33), the assumption $\langle 1, \psi_0 \rangle = 0$ and the fact that u_+ is constant on ∂D , we find

$$\begin{aligned} \int_B |\operatorname{grad} u|^2 dx &\leq - \int_{\Omega_r} u_- \frac{\partial u_-}{\partial \nu} ds - \int_{\partial D} u_+ \frac{\partial u_+}{\partial \nu} ds \\ &= - \int_{\Omega_r} u_- \frac{\partial u_-}{\partial \nu} ds + \int_{\partial D} u_+ \psi_0 ds = - \int_{\Omega_r} u_- \frac{\partial u_+}{\partial \nu} ds \end{aligned}$$

where Ω_r denotes a sphere with sufficiently large radius r centered at the origin (and interior normal ν). With the help of $\int_{\partial D} \psi_0 ds = 0$, using (6.14) and (6.15), it can be seen that u has the asymptotic behavior (6.16) with $u_\infty = 0$. Therefore, passing to the limit $r \rightarrow \infty$, we arrive at the contradiction $\int_B |\operatorname{grad} u|^2 dx \leq 0$. Hence, u is constant in $\mathbb{R}^m \setminus \bar{D}$ and from the jump relation (6.33) we derive the contradiction $\psi_0 = 0$. Therefore, $\langle 1, \psi_0 \rangle \neq 0$. The statement on the Riesz number is a consequence of Problem 4.4. \square

Theorem 6.21 *The double-layer potential*

$$u(x) = \int_{\partial D} \varphi(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)} ds(y), \quad x \in D, \quad (6.34)$$

with continuous density φ is a solution of the interior Dirichlet problem provided that φ is a solution of the integral equation

$$\varphi(x) - 2 \int_{\partial D} \varphi(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)} ds(y) = -2f(x), \quad x \in \partial D. \quad (6.35)$$

Proof. This follows from Theorem 6.17. \square

Theorem 6.22 *The interior Dirichlet problem has a unique solution.*

Proof. The integral equation $\varphi - K\varphi = -2f$ of the interior Dirichlet problem is uniquely solvable by Theorem 3.4, since $N(I - K) = \{0\}$. \square

From Theorem 6.14 we see that in order to obtain an integral equation of the second kind for the Dirichlet problem it is crucial to seek the solution in the form of a double-layer potential rather than a single-layer potential, which would lead to an integral equation of the first kind. Historically, this important observation goes back to Beir [14].

The double-layer potential approach (6.34) for the exterior Dirichlet problem leads to the integral equation $\varphi + K\varphi = 2f$ for the density φ . Since $N(I + K') = \operatorname{span}\{\psi_0\}$, by the Fredholm alternative, this equation is solvable if and only if $\langle f, \psi_0 \rangle = 0$. Of course, for arbitrary boundary data f we cannot expect this condition to be satisfied. Therefore we modify our approach as follows.

Theorem 6.23 *The modified double-layer potential*

$$u(x) = \int_{\partial D} \varphi(y) \left\{ \frac{\partial \Phi(x, y)}{\partial \nu(y)} + \frac{1}{|x|^{m-2}} \right\} ds(y), \quad x \in \mathbb{R}^m \setminus \bar{D}, \quad (6.36)$$

with continuous density φ is a solution to the exterior Dirichlet problem provided that φ is a solution of the integral equation

$$\varphi(x) + 2 \int_{\partial D} \varphi(y) \left\{ \frac{\partial \Phi(x, y)}{\partial \nu(y)} + \frac{1}{|x|^{m-2}} \right\} ds(y) = 2f(x), \quad x \in \partial D. \quad (6.37)$$

Here, we assume that the origin is contained in D . \square

Proof. This again follows from Theorem 6.17. Observe that u has the required behavior for $|x| \rightarrow \infty$, namely, $u(x) = O(1)$ if $m = 2$ and $u(x) = o(1)$ if $m = 3$. \square

Theorem 6.24 *The exterior Dirichlet problem has a unique solution.*

Proof. The integral operator $\tilde{K} : C(\partial D) \rightarrow C(\partial D)$ defined by

$$(\tilde{K}\varphi)(x) := 2 \int_{\partial D} \varphi(y) \left\{ \frac{\partial \Phi(x, y)}{\partial \nu(y)} + \frac{1}{|x|^{m-2}} \right\} ds(y), \quad x \in \partial D,$$

is compact, since the difference $\tilde{K} - K$ has a continuous kernel. Let φ be a solution to the homogeneous equation $\varphi + \tilde{K}\varphi = 0$ and define u by (6.36). Then $2u = \tilde{K}\varphi + \varphi = 0$ on ∂D , and by the uniqueness for the exterior Dirichlet problem it follows that $u = 0$ in $\mathbb{R}^m \setminus \bar{D}$. Using (6.15), we deduce the asymptotic behavior

$$|x|^{m-2}u(x) = \int_{\partial D} \varphi ds + O\left(\frac{1}{|x|}\right), \quad |x| \rightarrow \infty,$$

uniformly for all directions. From this, since $u = 0$ in $\mathbb{R}^m \setminus \bar{D}$, we obtain $\int_{\partial D} \varphi ds = 0$. Therefore $\varphi + K\varphi = 0$, and from Theorem 6.20 we conclude that φ is constant on ∂D . Now $\int_{\partial D} \varphi ds = 0$ implies that $\varphi = 0$, and the existence of a unique solution to the integral equation (6.37) follows from Theorem 3.4. \square

Theorem 6.25 *The single-layer potential*

$$u(x) = \int_{\partial D} \psi(y) \Phi(x, y) ds(y), \quad x \in D, \quad (6.38)$$

with continuous density ψ is a solution of the interior Neumann problem provided that ψ is a solution of the integral equation

$$\psi(x) + 2 \int_{\partial D} \psi(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)} ds(y) = 2g(x), \quad x \in \partial D. \quad (6.39)$$

Proof. This follows from Theorem 6.18. \square

Theorem 6.26 *The interior Neumann problem is solvable if and only if*

$$\int_{\partial D} g ds = 0 \quad (6.40)$$

is satisfied.

Proof. The necessity of condition (6.40) is a consequence of Green's theorem (6.3) applied to a solution u . The sufficiency of condition (6.40) follows from the fact that by Theorem 6.20 it coincides with the solvability condition of the Fredholm alternative for the inhomogeneous integral equation (6.39), i.e., for $\varphi + K'\psi = 2g$. \square

Theorem 6.27 *The single-layer potential*

$$u(x) = \int_{\partial D} \psi(y) \Phi(x, y) ds(y), \quad x \in \mathbb{R}^m \setminus \bar{D}, \quad (6.41)$$

with continuous density ψ is a solution of the exterior Neumann problem provided that ψ is a solution of the integral equation

$$\psi(x) - 2 \int_{\partial D} \psi(y) \frac{\partial \Phi(x, y)}{\partial \nu(x)} ds(y) = -2g(x), \quad x \in \partial D, \quad (6.42)$$

and, if $m = 2$, also satisfies

$$\int_{\partial D} \psi ds = 0. \quad (6.43)$$

Proof. Again this follows from Theorem 6.18. Observe that for $m = 2$ the additional condition (6.43) ensures that u has the required behavior $u(x) = o(1), |x| \rightarrow \infty$, as can be seen from (6.14). \square

Theorem 6.28 *In \mathbb{R}^3 the exterior Neumann problem has a unique solution. In \mathbb{R}^2 the exterior Neumann problem is uniquely solvable if and only if*

$$\int_{\partial D} g ds = 0 \quad (6.44)$$

is satisfied.

Proof. By Theorems 3.4 and 6.20 the equation $\psi - K'\psi = -2g$ is uniquely solvable for each right-hand side g . If (6.44) is satisfied, using the fact that $1 + K1 = 0$, we find

$$2\langle 1, \psi \rangle = \langle 1 - K1, \psi \rangle = \langle 1, \psi - K'\psi \rangle = -2\langle 1, g \rangle = 0.$$

Hence, the additional property (6.43) is satisfied in \mathbb{R}^2 . That condition (6.44) is necessary for the solvability in \mathbb{R}^2 follows from (6.9). \square

We finally show that the solutions depend continuously on the given boundary data.

Theorem 6.29 *The solutions to the Dirichlet and Neumann problems depend continuously in the maximum norm on the given data.*

Proof. For the Dirichlet problem the assertion follows from the maximum principle (Theorem 6.8). In two dimensions, for the exterior problem, from the form (6.36) of the solution u we observe that we have to incorporate the value u_∞ at infinity through $\int_{\partial D} \varphi ds$. But this integral depends continuously on the given boundary data, since the inverse $(I + \tilde{K})^{-1}$ of $I + \tilde{K}$ is bounded by Theorem 3.4.

For the Neumann problem we first observe that for single-layer potentials u with continuous density ψ for any closed ball B in \mathbb{R}^m we have an estimate of the form

$$\|u\|_{\infty, B} \leq \|w\|_{\infty, B} \|\psi\|_{\infty, \partial D},$$

where the function

$$w(x) := \int_{\partial D} |\Phi(x, y)| ds(y), \quad x \in \mathbb{R}^m,$$

is continuous in \mathbb{R}^m by Theorem 6.14. Then for the exterior problem choose a sufficiently large ball B and the continuous dependence of the solution on the boundary data in B follows from the boundedness of the inverse $(I - K')^{-1}$ of $I - K'$. In the remaining exterior of B , continuity then follows from the maximum-minimum principle.

For the interior problem we can expect continuity only after making the solution u unique by an additional condition, for example, by requiring that $\int_{\partial D} u ds = 0$. From $\langle 1, K'\psi \rangle = \langle K1, \psi \rangle = -\langle 1, \psi \rangle$ we observe that K' maps the closed subspace $C_0(\partial D) := \{\psi \in C(\partial D) : \int_{\partial D} \psi ds = 0\}$ into itself. By Theorem 6.20 the operator $I + K'$ has a trivial nullspace in $C_0(\partial D)$. Hence, the inverse $(I + K')^{-1}$ is bounded from $C_0(\partial D)$ onto $C_0(\partial D)$, i.e., the unique solution ψ_0 of $\psi_0 + K'\psi_0 = g$ satisfying $\int_{\partial D} \psi_0 ds = 0$ depends continuously on g . Therefore, as above, the corresponding single-layer potential u_0 depends continuously on g in the maximum norm. Finally, $u := u_0 - \int_{\partial D} u_0 ds / |\partial D|$ yields a solution vanishing in the integral mean on the boundary, and it depends continuously on g . \square

6.5 Nonsmooth Boundaries

Despite the fact that the integral equation method provides an elegant approach to constructively prove the existence of solutions for the boundary value problems of potential theory we do not want to disguise its major drawback: the relatively strong regularity assumption on the boundary to be of class C^2 . It is possible to slightly weaken the regularity and allow *Ljapunov boundaries* instead of C^2 boundaries and still remain within the framework of compact operators. The boundary is said to satisfy a Ljapunov condition if at each point $x \in \partial D$ the normal vector ν exists and there are positive constants L and α such that for the angle $\vartheta(x, y)$ between the normal vectors at x and y the estimate $\vartheta(x, y) \leq L|x - y|^\alpha$ holds for all $x, y \in \partial D$. For the treatment of the Dirichlet and Neumann problem for Ljapunov boundaries, which does not differ essentially from that for C^2 boundaries, we refer to [123].

However, the situation changes considerably if the boundary is allowed to have edges and corners. This effects the form of the integral equations

and the compactness of the integral operators as we will demonstrate by considering the interior Dirichlet problem in a two-dimensional domain D with corners. We assume that the boundary ∂D is piecewise twice differentiable, i.e., ∂D consists of a finite number of closed arcs $\Gamma_1, \dots, \Gamma_p$ that are all of class C^2 and that intersect only at the corners x_1, \dots, x_p . At the corners the normal vector is discontinuous (see Fig. 6.1 for a domain with three corners).

For simplicity, we restrict our analysis to boundaries that are straight lines in a neighborhood of each of the corners. In particular, this includes the case where ∂D is a polygon. The interior angle at the corner x_i we denote by γ_i and assume that $0 < \gamma_i < 2\pi$, $i = 1, \dots, p$, i.e., we exclude cusps. For a boundary with corners, the continuity of the double-layer potential with continuous density as stated in Theorem 6.17 remains valid, but at the corners the jump relation (6.24) has to be modified into the form

$$v_{\pm}(x_i) = \int_{\partial D} \varphi(y) \frac{\partial \Phi(x_i; y)}{\partial \nu(y)} ds(y) \pm \frac{1}{2} \delta_i^{\pm} \varphi(x_i), \quad i = 1, \dots, p, \quad (6.45)$$

where $\delta_i^+ = \gamma_i/\pi$ and $\delta_i^- = 2 - \gamma_i/\pi$. It is a matter of straightforward application of Green's theorem as in Example 6.16 to verify (6.45) for constant densities. For arbitrary continuous densities, the result can be obtained from the C^2 case of Theorem 6.17 by a superposition of two double-layer potentials on two C^2 curves intersecting at the corner with the density φ equal to zero on the parts of the two curves lying outside ∂D .

Trying to find the solution to the interior Dirichlet problem in the form of a double-layer potential with continuous density φ as in Theorem 6.21 reduces the boundary value problem to solving the integral equation $\varphi - \tilde{K}\varphi = -2f$, where the operator $\tilde{K} : C(\partial D) \rightarrow C(\partial D)$ is given by

$$(K\varphi)(x) := \begin{cases} (K\varphi)(x) + \left(\frac{\gamma_i}{\pi} - 1\right)\varphi(x_i), & x = x_i, i = 1, \dots, p. \\ \end{cases}$$

Note that for $\varphi \in C(\partial D)$, in general, $K\varphi$ is not continuous at the corners. However $\tilde{K}\varphi$ is continuous, since it is the sum $\tilde{K}\varphi = v_+ + v_-$ of the continuous boundary values of the double-layer potential v . By Problem 6.1 the kernel

$$k(x, y) := \frac{\nu(y) \cdot \{x - y\}}{\pi|x - y|^2}.$$

of the integral operator K is continuous on $\Gamma_i \times \Gamma_i$ for $i = 1, \dots, p$. Singularities of the kernel occur when x and y approach a corner on the two different arcs intersecting at the corner.

For $n \in \mathbb{N}$ we use the continuous cutoff function h introduced in the proof of Theorem 2.22 to define the operators $K_n : C(\partial D) \rightarrow C(\partial D)$ by

$$(K_n\varphi)(x) := \int_{\partial D} h(n|x - y|) k(x, y) \varphi(y) ds(y), \quad x \in \partial D.$$

For each $n \in \mathbb{N}$ the operator K_n is compact, since its kernel is continuous on $\partial D \times \Gamma_i$ for $i = 1, \dots, p$, i.e., we can interpret K_n as the sum of p integral operators with continuous kernels on $\partial D \times \Gamma_i$ by subdividing the integral over ∂D into a sum of integrals over the arcs Γ_i for $i = 1, \dots, p$.

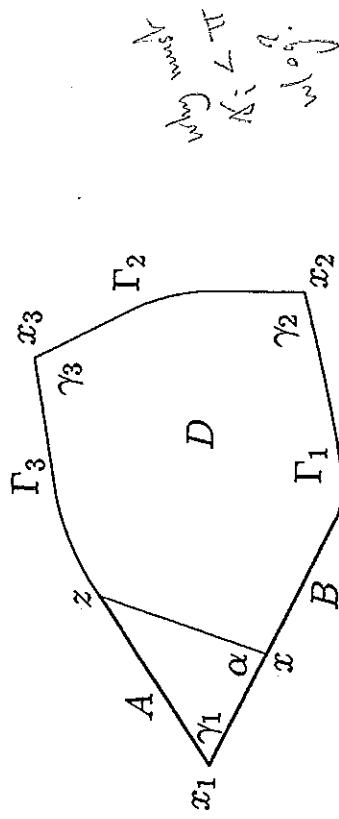


FIGURE 6.1. Domain with a corner

Now consider $\tilde{K}_n := \tilde{K} - K_n$ and assume that n is large enough that for each $x \in \partial D$ the disk $B[x; 1/n] = \{y \in \mathbb{R}^2 : |x - y| \leq 1/n\}$ intersects only either one or, in the vicinity of the corners, two of the arcs Γ_i . By our assumption on the nature of the corners we can assume n is large enough that in the second case the intersection consists of two straight lines A and B (see Fig. 6.1). Let

$$M := \max_{i=1, \dots, p} \max_{x, y \in \Gamma_i} |k(x, y)|.$$

Then, by projection onto the tangent line, for the first case we can estimate

$$|(\tilde{K}_n\varphi)(x)| \leq M \|\varphi\|_{\infty} \int_{\partial D \cap B[x; 1/n]} ds(y) \leq M \|\varphi\|_{\infty} \frac{4}{\pi} \int_{\Gamma_i} ds(y).$$

In the second case, we first note that for $x \in B \setminus \{x_i\}$, by Green's theorem (6.3) applied in the triangle with the corners at x and at the endpoints x_i and z of A we have

$$\int_A \left| \frac{\partial \Phi(x, y)}{\partial \nu(y)} \right| ds(y) = \left| \int_A \frac{\partial \Phi(x, y)}{\partial \nu(y)} ds(y) \right| = \left| \frac{\alpha(x)}{2\pi} \right|,$$

where $\alpha(x)$ denotes the angle of this triangle at the corner x (see Fig. 6.1). Elementary triangle geometry shows that $\alpha(x) + \gamma_i \leq \pi$, where, without loss of generality, we have assumed that $\gamma_i < \pi$. Therefore, since for $x \in B \setminus \{x_i\}$ we have $k(x, y) = 0$ for all $y \in B \setminus \{x_i\}$, we obtain

$$|(\tilde{K}_n \varphi)(x)| \leq 2\|\varphi\|_\infty \left| \int_A \frac{\partial \Phi(x, y)}{\partial \nu(y)} ds(y) \right| \leq \frac{\alpha(x)}{\pi} \|\varphi\|_\infty \leq \left(1 - \frac{\gamma_i}{\pi}\right) \|\varphi\|_\infty.$$

Finally, for the corner x_i at the intersection of A and B we have

$$(\tilde{K}_n \varphi)(x_i) = \left(\frac{\gamma_i}{\pi} - 1\right) \varphi(x_i),$$

since $k(x_i, y) = 0$ for all $y \in (A \cup B) \setminus \{x_i\}$. Combining these results we observe that we can choose n large enough that $\|\tilde{K}_n\|_\infty \leq q$ where

$$q := \max_{i=1, \dots, p} \left|1 - \frac{\gamma_i}{\pi}\right| < 1.$$

Hence, we have a decomposition $I - \tilde{K} = I - \tilde{K}_n - K_n$, where $I - \tilde{K}_n$ has a bounded inverse by the Neumann series Theorem 2.9 and where K_n is compact. It is left to the reader to carry over the proof for injectivity of the operator $I - \tilde{K}$ from Theorem 6.20 to the case of a boundary with corners. Then existence of a solution to the inhomogeneous equation $\varphi - \tilde{K}\varphi = -2f$ follows by Corollary 3.6.

This idea of decomposing the integral operator into a compact operator and a bounded operator with norm less than one reflecting the behavior at the corners goes back to Radon [151] and can be extended to the general two-dimensional case and to three dimensions. For details we refer to Cryer [28], Král [94], and Wendland [182]. For a more comprehensive study of boundary value problems in domains with corners we refer to Grisvard [57]. For the integral equation method in Lipschitz domains we refer to Vechota [179].

Finally, we wish to mention that the integral equations for the Dirichlet and Neumann problems can also be treated in the space $L^2(\partial D)$ allowing boundary data in $L^2(\partial D)$. This requires the boundary conditions to be understood in a weak sense, which we want to illustrate by again considering the interior Dirichlet problem. We say that a harmonic function u in D assumes the boundary values $f \in L^2(\partial D)$ in the L^2 sense if

$$\lim_{h \rightarrow +0} \int_{\partial D} [u(x - h\nu(x)) - f(x)]^2 ds(x) = 0.$$

To establish uniqueness under this weaker boundary condition, we choose parallel surfaces $\partial D_h := \{x - h\nu(x) : x \in \partial D\}$ to ∂D with $h > 0$ sufficiently small. Then, following Miranda [125], for

$$J(h) := \int_{\partial D_h} u^2 ds, \quad h > 0,$$

we can write

$$J(h) = \int_{\partial D} \{1 + 2hH(x) + h^2 K(x)\} [u(x - h\nu(x))]^2 ds(x)$$

and differentiate to obtain

$$\frac{1}{2} \frac{dJ}{dh} = - \int_{\partial D_h} u \frac{\partial u}{\partial \nu} ds + \int_{\partial D} \{H(x) + hK(x)\} [u(x - h\nu(x))]^2 ds(x).$$

Hence, using Green's theorem (6.1), we have

$$\frac{1}{2} \frac{dJ}{dh} = - \int_{D_h} |\operatorname{grad} u|^2 dx + \int_{\partial D} \{H + hK\} [u \cdot -h\nu]^2 ds, \quad (6.46)$$

where D_h denotes the interior of the parallel surface ∂D_h . Now let u vanish on the boundary ∂D in the L^2 sense and assume that $\operatorname{grad} u \neq 0$ in D . Then there exists some closed ball B contained in D such that $I := \int_B |\operatorname{grad} u|^2 dx > 0$, and from (6.46) we deduce that $dJ/dh \leq -I$ for all $0 < h \leq h_0$ and some sufficiently small $h_0 > 0$. Since J is continuous on $[0, h_0]$, is continuously differentiable on $(0, h_0]$, and satisfies $J(0) = 0$, we see that $J(h) \leq -Ih$ for all $0 < h \leq h_0$. This is a contradiction to $J(h) \geq 0$ for all $h > 0$. Therefore u must be constant in D , and from $J(0) = 0$ we obtain $u = 0$ in D .

Using the fact that, due to Theorem 6.6, on the parallel surfaces ∂D_h for $h > 0$ there is more regularity of u , a different approach to establishing uniqueness under weaker assumptions was suggested by Calderón [19]. It is based on representing u in terms of the double-layer operator K_h on ∂D_h and establishing $\|K_h - K\|_{L^2(\partial D)} \rightarrow 0$ as $h \rightarrow 0$.

To prove existence of a solution for boundary conditions in the L^2 sense via the surface potential approach, it is necessary to extend the jump relations of Theorem 6.14, 6.17, 6.18, and 6.19 from $C(\partial D)$ onto $L^2(\partial D)$. This can be achieved quite elegantly through the use of Lax's Theorem 4.11 as worked out by Kersten [83]. In particular, for the double-layer potential v with density $\varphi \in L^2(\partial D)$, the jump relation (6.24) has to be replaced by

$$\lim_{h \rightarrow +0} \int_{\partial D} [2v(x \pm h\nu(x)) - (K\varphi)(x) \mp \varphi(x)]^2 ds(x) = 0. \quad (6.47)$$

From this, we see that the double-layer potential with density $\varphi \in L^2(\partial D)$ solves the Dirichlet problem with boundary values $f \in L^2(\partial D)$ provided the density solves the integral equation $\varphi - K\varphi = -2f$ in the space $L^2(\partial D)$. Noting that integral operators with weakly singular kernels are compact from $L^2(\partial D)$ into $L^2(\partial D)$ (see Problem 4.5), for existence and uniqueness of a solution to this integral equation we need to establish that the homogeneous equation admits only the trivial solution. From Theorem 6.20 we know that the operator $I - K$ has a trivial nullspace in $C(\partial D)$. Therefore,

by the Fredholm alternative applied in the dual system $\langle C(\partial D), L^2(\partial D) \rangle$ with the L^2 bilinear form, the adjoint operator $I - K'$ has a trivial nullspace in $L^2(\partial D)$. Again by the Fredholm alternative, but now applied in the dual system $\langle L^2(\partial D), L^2(\partial D) \rangle$ with the L^2 bilinear form, the operator $I - K$ has a trivial nullspace in $L^2(\partial D)$. This idea to use the Fredholm alternative in two different dual systems for showing that the nullspaces for weakly singular integral operators of the second kind in the space of continuous functions and in the L^2 space coincide is due to Hähner [64].

We will come back to the potential theoretic boundary value problems in the next two chapters. In Section 7.5 we will solve the exterior Dirichlet and Neumann problems in two dimensions by integral equations of the first kind in a Hölder space setting. And in Section 8.3, again in two dimensions, we will solve the integral equations of the second kind in Sobolev spaces leading to weak solutions of the boundary value problems.

For integral equation methods for boundary value problems for the Helmholtz equation $\Delta u + k^2 u = 0$, i.e., for acoustic and electromagnetic scattering problems, we refer to [24, 25].

Problems

6.1 Use a regular 2π -periodic parameterization $\partial D = \{x(t) : 0 \leq t \leq 2\pi\}$ with counterclockwise orientation for the boundary curve to transform the integral equation (6.35) of the interior two-dimensional Dirichlet problem into the form

$$\tilde{\varphi}(t) - \int_0^{2\pi} k(t, \tau) \tilde{\varphi}(\tau) d\tau = -2\tilde{f}(t), \quad 0 \leq t \leq 2\pi,$$

where $\tilde{\varphi}(t) := \varphi(x(t))$, $\tilde{f}(t) := f(x(t))$ and the kernel is given by

$$k(t, \tau) = \begin{cases} \frac{1}{\pi} \frac{[x'(\tau)]^\perp \cdot [x(t) - x(\tau)]}{|x(t) - x(\tau)|^2}, & t \neq \tau, \\ \frac{1}{2\pi} \frac{[x'(t)]^\perp \cdot x''(t)}{|x'(t)|^2}, & t = \tau, \end{cases}$$

where $[x']^\perp := (x'_2, -x'_1)$. Show that this kernel is continuous provided ∂D is of class C^2 .

6.2 Show that for an ellipse with parametric representation $x_1(t) = a \cos t$, $x_2(t) = b \sin t$, the kernel k of Problem 6.1 is given by

$$k(t, \tau) = -\frac{ab}{\pi} \frac{1}{a^2 + b^2 - (a^2 - b^2) \cos(t + \tau)}.$$

6.3 Extend Theorem 6.20 to domains D with nonconnected boundaries and, in particular, show that $\dim N(I - K) = p$, where p denotes the number of bounded components of $\mathbb{R}^m \setminus \bar{D}$. For the interior Dirichlet problem establish existence of a solution through a modification of the integral equation (6.35) analogous to (6.36) by adding a point source in each of the bounded components of $\mathbb{R}^m \setminus \bar{D}$.

6.4 Let $D \subset \mathbb{R}^2$ be of class C^2 and strictly convex in the sense that the curvature of the boundary ∂D is strictly positive. Show that there exists a constant $0 < \delta < 1$ such that

$$\int_{\partial D} \left| \frac{\partial \Phi(x_1, y)}{\partial \nu(y)} - \frac{\partial \Phi(x_2, y)}{\partial \nu(y)} \right| ds(y) \leq 1 - \delta$$

for all $x_1, x_2 \in \partial D$.

Hint: Use Example 6.16, Problem 6.1, and the property that

$$\frac{\partial \Phi(x, y)}{\partial \nu(y)} = \frac{\nu(y) \cdot \{x - y\}}{2\pi|x - y|^2}$$

is negative on $\partial D \times \partial D$ to verify that

$$\left| \int_D \left\{ \frac{\partial \Phi(x_1, y)}{\partial \nu(y)} - \frac{\partial \Phi(x_2, y)}{\partial \nu(y)} \right\} ds(y) \right| \leq \frac{1}{2} - a|\partial D|,$$

for each Jordan measurable subset $\Gamma \subset \partial D$, where

$$a := \min_{x, y \in \partial D} \left| \frac{\partial \Phi(x, y)}{\partial \nu(y)} \right| > 0.$$

6.5 In 1870 Neumann [137] gave the first rigorous proof for the existence of a solution to the two-dimensional interior Dirichlet problem in a strictly convex domain of class C^2 . By completely elementary means he established that the successive approximations

$$\varphi_{n+1} := \frac{1}{2} \varphi_n + \frac{1}{2} K \varphi_n - f, \quad n = 0, 1, 2, \dots,$$

with arbitrary $\varphi_0 \notin C(\partial D)$ converge uniformly to the unique solution φ of the integral equation $\varphi - K\varphi = -2f$. In functional analytic terms his proof amounted to showing that the operator L given by $L := \frac{1}{2}(I + K)$ is a contraction with respect to the norm

$$\|\varphi\| := \left| \sup_{z \in \partial D} \varphi(z) - \inf_{z \in \partial D} \varphi(z) \right| + \alpha \sup_{z \in \partial D} |\varphi(z)|,$$

where $\alpha > 0$ is appropriately chosen. This norm is equivalent to the maximum norm. Derive the above results for yourself.

Hint: Use Problem 6.4 to show that $\|L\| \leq (2 - \delta + \alpha)/2$ by writing

$$(L\varphi)(x) = \int_{\partial D} [\varphi(y) - \varphi(x)] \frac{\partial \Phi(x, y)}{\partial \nu(y)} ds(y)$$

and

$$(L\varphi)(x_1) - (L\varphi)(x_2) = \frac{1}{2} [\varphi(x_1) - \varphi(x)] - \frac{1}{2} [\varphi(x_2) - \varphi(x)] \\ + \int_{\partial D} [\varphi(y) - \varphi(x)] \left\{ \frac{\partial \Phi(x_1, y)}{\partial \nu(y)} - \frac{\partial \Phi(x_2, y)}{\partial \nu(y)} \right\} ds(y),$$

where $x \in \partial D$ is chosen such that $\varphi(x) = \{\sup_{z \in \partial D} \varphi(z) + \inf_{z \in \partial D} \varphi(z)\}/2$ (see [81]).