M126 Lec 12. ① 2/14/12.

OH Tu 2-3.

• Prove JR3 from Lec 11 Discuss project.txt. HW5: Neu int?

HW5: convergence rate worse nr. $\partial\Omega$
error research prob. how detoriates!

Other BVPs.

We did interior Dirichlet : "what temperature dist does uniformly conducting body settle to when bdry values set to func. f ?"

• Interior Neumann: $\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u_n = f & \text{on } \partial\Omega \end{cases}$ equilibrium temp. distn, f = specified heat input flux at each pt on $\partial\Omega$.

What if pumping in more heat than extracting? blow up!

Already know u harm. $\Rightarrow$ ZF: $\int_{\partial\Omega} u_n = 0$ ie $\int_{\partial\Omega} f = 0$ for existence.

Then if u_1, u_2 are solns, $w = u_1 - u_2$ sat $\begin{Bmatrix} \Delta w = 0 & \text{in } \Omega \\ w_n = 0 & \text{on } \partial\Omega \end{Bmatrix}$ w = const. a soln. [the only soln: GI1: $\int w \underbrace{\Delta w}_{0} - \nabla w \cdot \nabla w = \int w \underbrace{w_n}_{0}$ so $\nabla w \equiv 0$.

$\Rightarrow$ when exists, soln. is unique only up to a const. (Δ op has $\lambda = 0$ a Neumann eigenvalue).

BIE solution? if use $u = \mathcal{D}\tau$ as before, BC is $\underbrace{\int_{\partial\Omega} \frac{\partial^2 \Phi}{\partial n_x \partial n_y}(x,y)\, \tau(y)\, ds_y}_{\text{deriv of DLP: } T\tau} = f$ hypersingular: kernel $\sim \frac{1}{|x-y|^2}$ nr diag not integrable! unbounded op.

instead try? $u = \mathcal{S}\sigma$ so $\underbrace{\int_{\partial\Omega} \frac{\partial \Phi}{\partial n_x}(x,y)\, \sigma(y)\, ds_y}_{D^T\sigma} + \frac{\sigma(x)}{2} \overset{\text{JR2}}{=} u_n^- = f$

IE is $(I + 2D^T)\sigma = 2f$. 2^{nd} kind again. D cpt $\Leftrightarrow D^T$ cpt.

But we know nonunique since BVP is, but in practice, backwards-stable lin. solver should give
Got here. can prove via ext. GRF. $\to$ cond # of matrix $\to \infty$. a $\vec{\sigma}$ which satisfies.
Danger: the const will be large $\Rightarrow$ loss of digits in $\vec{\sigma}$.

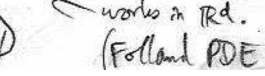
Atkinson book: solve $(I + 2D^T)\sigma + \sigma(x_0) = 2f$
(p. 336–7).

where $x_0 \in \partial\Omega$ fixed.
Proves uniquely solvable $\forall f \in C(\partial\Omega)$.

• Exterior BVPs: eg Dirichlet (ED) $\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2 \setminus \bar{\Omega} \\ u = f & \text{on } \partial\Omega \\ u(x) = O(1) & \text{as } |x| \to \infty \quad \text{ie bounded.} \end{cases}$ (harmonic at ∞)

has unique soln $\forall f \in C(\partial\Omega)$ for $d > 2$ this is $o(1)$. $\longrightarrow$ physically this needed since potential at ∞ needs to be specified.

Follows from $\tilde{u}(x) := u\left(\frac{x}{|x|^2}\right)$ "Kelvin xform of u" maps $\mathbb{R}^2 \setminus \bar{\Omega}$ to $\tilde{\Omega}$ $\leftarrow$ works in $\mathbb{R}^d$.

$\tilde{u}$ is harmonic in $\tilde{\Omega} = \{x : \frac{x}{|x|^2} \in \mathbb{R}^2 \setminus \bar{\Omega}\}$ ($d=2$ only!)  (Folland PDE book)

Condition at ∞ mapped to 'analytic at 0'