M126 lec 14 — finish Helmholtz · start fast algorithms. · Mike O'Neil talk 1-2pm. · show {dlp-eigvalsweep.m, Neumann eigfuncs., lowrank-curve.m} ① Th. 2/21/12.

• Where do Hankel funcs come from? want $\underbrace{(\Delta_x + k^2)}_{\text{Helm op.}}\Phi(x,y) = 0$ for $\forall x \neq y$.

$d=2$: wlog $y=0$. call $u := \Phi(\cdot,0)$, want sat. Helm. eqn.

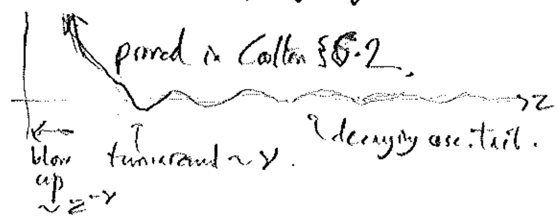
$u(r,\theta) = f(kr)e^{i\nu\theta}$ polar sep. of var., $\nu \in \mathbb{Z}$ so single-valued, solve for f:

$0 = (\Delta + k^2)u = \underbrace{\frac{1}{r}\partial_r(r\partial_r u) + \frac{1}{r^2}\partial_{\theta\theta}u}_{\text{laplacian}} + k^2 u = (k^2 f'' + \frac{k}{r}f')e^{i\nu\theta} + \frac{(i\nu)^2}{r^2}fe^{i\nu\theta} + k^2 f e^{i\nu\theta}$ (chain rule.)

cancel $e^{i\nu\theta}$, gather $kr = z$: (& mult. by r^2) $z^2 f'' + zf' + (z^2 - \nu^2)f = 0$ Bessel's eqn., order ν (ODE)

$H_\nu^{(1)}(z)$ is soln. w/ log singular @ $z \to 0^+$

large-argument: $z \to \infty$ $H_\nu^{(1)}(z) = \sqrt{\frac{2}{\pi z}}\, e^{i(z - \frac{\nu\pi}{2} - \frac{\pi}{4})} + O(\frac{1}{z})$ proved in Colton §6.2



Are also solutions regular at $z=0$: $J_\nu(z)$ Bessel funcs.

Fixing nonuniqueness in BIE for scattering.

In HW6 you saw ext Dir BIE haunted by ghost of complementary BVP: let $u^s = \mathcal{D}\tau$, sat. Helm in $\mathbb{R}^d\setminus\bar{\Omega}$, radiative, solves ext BVP if (BIE): $\underbrace{(I + 2D)\tau}_{2u^s \text{ by JR3.}} = 2f = -2u^i|_{\partial\Omega}$ from inc. field. Haunting is: $I + 2D$ singular for certain set of k → cond'n-ios, no soln. for some RHS's.

• We'll need GRF (interior), same as Laplace: same log.sing. {Let $(\Delta + k^2)u = 0$ in Ω, then $\mathcal{S}u_n - \mathcal{D}u|_{\partial\Omega} = \begin{cases} u & \text{in } \Omega \\ 0 & \text{in } \mathbb{R}^d\setminus\bar{\Omega}\end{cases}$

Suppose $\phi \not\equiv 0$ sat. $\begin{cases}(\Delta + k^2)\phi = 0 & \text{in } \Omega \\ \phi_n = 0 & \text{on } \partial\Omega\end{cases}$ then ϕ is interior Neumann eigenfunc. & k^2 its eigenvalue. ("acoustic resonance of cavity" Ω).

then by GRF, $\mathcal{S}\underset{0}{\phi_n} - \mathcal{D}\phi|_{\partial\Omega} = \phi$ in Ω.

take $x \to \partial\Omega^-$ & use JR3: $-(D - \frac{1}{2})\phi|_{\partial\Omega} = \phi|_{\partial\Omega}$ ie $(I + 2D)\phi|_{\partial\Omega} = 0$.

since $\phi|_{\partial\Omega}$ nontriv. (otherwise $\phi \equiv 0$ by GRF), $\dim \text{Nul}(I + 2D) > 0$, singular, not soluble $\forall f$! (Fred. Alt. (just like sq. matrix))

Show evolving eigval of $2D$ vs k: when hit $\begin{cases}-1 \\ +1\end{cases}$ $k^2 = \begin{cases}\text{Neu} \\ \text{Dir}\end{cases}$ eigval of Ω

↳ project: use smooth evolution to find such k^2's.

Fix it: rep. $u^s = (\mathcal{D} - i\eta\mathcal{S})\tau$, $\eta > 0$ Brakhage–Werner, Leis, Panich, '60s,

solves ext Dir BVP if $(\underset{\text{JR3 as before}}{I + 2D} - \underset{\text{JR1 (no jump for S val.)}}{2i\eta S})\tau = 2f$

Thm: $I + 2D - 2i\eta S$ injective $\forall k > 0$

Pf: let τ solve $(\frac{1}{2} + D - i\eta S)\tau = 0$, wish to show $\tau \equiv 0$.

from τ create potential $v := (\mathcal{D} - i\eta\mathcal{S})\tau$, then $v^+ = 0$ by construction of BIE (RHS $2f = 0$).

② 2/21/12

$\Rightarrow$ $v = 0$ in $\mathbb{R}^2 \setminus \overline{\Omega}$ by uniqueness of ext. Dir. BVP for radiative solns. ← a PDE result (Colton 56.5)

$\Rightarrow$ $v_n^+ = 0$ on $\partial\Omega$

$\Rightarrow$ JR1,3 $\Rightarrow$ $v^- = -\tau$
JR2,4 $\Rightarrow$ $v_n^- = -i\eta\tau$ } (a)

GII in Ω:
$$\underbrace{\int_{\partial\Omega} \bar{v}^- v_n^- \, ds}_{\text{by (a)}\; +i\eta \int_{\partial\Omega} |\tau|^2 ds} = \underbrace{\int_\Omega \bar{v}\Delta v + \nabla\bar{v}\cdot\nabla v \; dx}_{-k^2|v|^2 + |\nabla v|^2 \quad \text{pure real.}}$$

↑ but complex k messes this up.

Take Im part: $\tau \equiv 0$.
(& $\eta \neq 0$).

QED.

Notes:

i) Call such a scheme robust since provably never fails; similar exist for Neu ext BVP, transmission, etc. Maxwell (ACMS!)

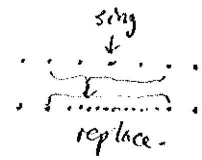
ii) Quadrature of BIE now harder: S has log singularity

Approaches: a) use 'correction' of periodic trap. rule weights near diagonal so omits diagonal $i=j$ & integrates smooth + $\log|s-t| \cdot$ smooth to high order.
Kapur - Rokhlin '97.

b) find exact weights to integrate log · smooth globally: product quadrature)
Kress '91, better but more analytic work.

projects, ⇐ research.

c) other ways to correct near singularity using new set of nodes
Alpert '99.



These also make $\mathcal{D}$ quadr. high order (HW6: noticed only 3rd order, unlike Laplace $k=0$ case was exponential).

Fast Algorithms: how people solve big problems.

eg $N = 10^6$: can't even fill Nyström matrix A ($10^{12} \times 16$ bytes $= 16000$ GB)
demanded by complex geom or 3d surface.

let alone do dense linear solve ($N^3 = 10^{18}$ flops)! $A\vec{x} = \vec{b}$

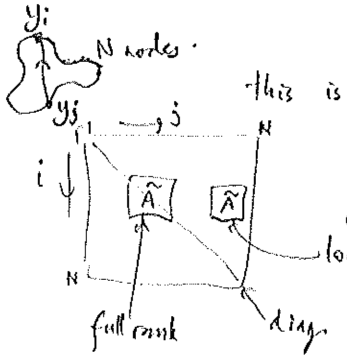
Instead: iterative methods. eg 'GMRES' (NLA Ch. 35), each iter. involves $\vec{x} \mapsto A\vec{x}$
converges, stop when residual error $\|A\vec{x} - \vec{b}\|$ small enough for you.

For well-conditioned 2nd kind IE, takes only $\underbrace{10-20}_{O(1), \text{ ie indep. of } N!}$ iters to get many digits (10^{-14}) accuracy.
But 1st kind terrible convergence rate, unless...

So now, whole scheme to solve for $\vec{\tau}$ is $O(N^2)$ since $x \mapsto Ax$ is.

Can we apply dense $N\times N$ ie Nyström matrix to a vector $\vec{x}$ faster than $O(N^2)$? Yes!

Toy problem $\begin{cases} \text{Let } y_i \in \mathbb{R}^2 \text{ be set of nodes.} \\ A \text{ has elements } a_{ij} = \begin{cases} \ln \frac{1}{|y_i - y_j|} & i \neq j \\ 0 & i = j \end{cases} \end{cases}$



this is off-diag part of Nyström matrix for S operator (Laplace), without weights w_j.

run lowrank_curve.m w/ $N = 1e3$, $5e3$ ↳ also apps to

numerical lowrank: small (~ 10) & indep. of N!

low rank requires source – target separation.

$\tilde{A}$ low num. rank means $\tilde{A} \approx PQ = N\big[\,\big]_{\sim 10}\ \big[\overset{N}{\quad\quad}\big]\sim 10$ eg via SVD, (but that's too slow in practice)

Fix an off-diag block, call it size $N \times N$: sources $y_j \in \mathbb{R}^2$, $j = 1 \ldots N$, targets $z_i \in \mathbb{R}^2$, $i = 1 \ldots N$.

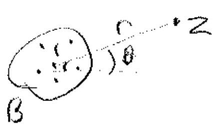
wish to compute $u_i = \sum_{j=1}^{N} x_j \ln \frac{1}{|z_i - y_j|} = (\tilde{A}\vec{x})_i$, $i = 1 \ldots N$.

x_j 'charge strength' at each node.

Potential due to sources $u(z) = \sum_{j=1}^{N} x_j \ln \frac{1}{|z - y_j|}$ harmonic for $z \neq y_j$ $j = 1 \ldots N$.

Goal is eval u @ targets z_i $i = 1 \ldots N$.

Thm (multipole expansion): outside a disc B centered at 0, containing all $\{y_j\}$, we can write



$$u(r,\theta) = \underbrace{c_0 \ln \frac{1}{r}}_{\text{monopole}} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)\, r^{-n}$$ ← Fourier series on each circle r = const

or writing $z = r e^{i\theta}$, $u(z) = \mathrm{Re}\left\{ c_0 \ln \frac{1}{z} + \underbrace{\sum_{n=1}^{\infty} c_n z^{-n}}_{\text{Laurent expansion}} \right\}$

sums abs. convergent in $\mathbb{R}^2 \setminus \overline{B}$