Lec (7) M126. ① 3/1/12.

Handling quadrature for singular kernels.

Recall Helmholtz BVP (e.g. for scattering) : need Nyström for $D - i\eta S$

cont. but not analytic ⇒ periodic trap. rule low order. — log singular on diag; PTR fails.

Are 'cheap' ways to correct PTR : Kapur-Rokhlin '97 changes weights near diag, sets diag $i=j$ to zero. Not v. good. (~ 50 nodes per wavelength needed for high acc.).

Best is 'product quadrature' (as in Kress '91): (~ 6 nodes per wavelength gets you 14 digits!). ① 3/1/12

Eg $$\int_0^{2\pi} f(s)\, g(s)\, ds \approx \sum_{j=1}^N w_j f(s_j)$$ where restrict $s_j = \frac{2\pi j}{N}$ i.e. PTR.

desired func. (real, smooth) — fixed 'weight' func (real, may be not smooth). — modified, not all just $\frac{2\pi}{N}$ for our g

Given g, how get $\{w_j\}_{j=1}^N$? Let's assume N even. (odd similar).

realize it's (f,g) & use Fourier series $f(s) = \sum_{n\in\mathbb{Z}} f_n e^{ins}$ (coeffs) $\leftrightarrow$ $f_n = \frac{1}{2\pi}\int_0^{2\pi} f(s) e^{-ins} ds$.

$$\left(\sum_n f_n e^{ins}, \sum_m g_m e^{ims}\right) = \sum_n \sum_m \bar{f}_n g_m \underbrace{\int_0^{2\pi} e^{-ins+ims} ds}_{=?\ \begin{cases}2\pi & n=m\\ 0 & \text{otherwise}\end{cases}}$$ $\{e^{ins}\}$ orthog. basis.

$$= 2\pi \sum_m \bar{f}_m g_m$$ Parseval. ($\ell_2(\mathbb{Z})$ inner-prod.)

If f smooth, $|f_m| \to 0$ fast as $|m| \to \infty$.

In particular, if f analytic in strip $|\mathrm{Im}\, s| \le \alpha$ & bounded, then $f_m = O(e^{-\alpha|m|})$. Ex: prove this.

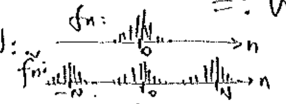
Use usual PTR to approx coeff formula: $$f_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-ins} f(s) ds \approx \tilde{f}_n := \frac{1}{N}\sum_{j=1}^N e^{-ins_j} f(s_j) \qquad (*)$$

Why good? sub. F. series for f: $$\tilde{f}_n = \frac{1}{N}\sum_{j=1}^N e^{-ins_j} \sum_{m\in\mathbb{Z}} f_m e^{ims_j} = \sum_{m\in\mathbb{Z}} f_m \underbrace{\frac{1}{N}\sum_{j=1}^N e^{i(m-n)\frac{2\pi j}{N}}}_{\begin{cases}1 & m=n \pmod N\\ 0 & \text{otherwise.}\end{cases}}$$

so $\tilde{f}_n = f_n + \underbrace{f_{n+N} + f_{n-N} + f_{n+2N} + f_{n-2N} + \cdots}_{\text{aliasing error: small if F. coeffs decay rapidly.}}$ So (*) exact for $\{e^{ins}\}_{|n|<N/2}$

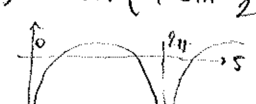
then $$\int_0^{2\pi} f(s) g(s) ds = 2\pi \sum_{n\in\mathbb{Z}} \bar{f}_n g_n \approx 2\pi {\sum_n}' \bar{f}_n g_n \underset{(*)}{\approx} \frac{2\pi}{N}\sum_{j=1}^N f(s_j) \underbrace{{\sum_n}' e^{ins_j} g_n}_{=:\, W_j}$$

since f_n exp small for $|n| > N/2$. — meaning $n \in [-N/2, N/2]$ but weighting ends by 1/2. — why truncate? since $\tilde{f}_n$ repeats @ N:



So $$w_j = \frac{2\pi}{N} {\sum_n}' g_n e^{ins_j} \longrightarrow e^{\frac{2\pi i j n}{N}}$$ i.e. $\{w_j\}$ = size N DFT of first N Fourier coeffs of g.

E.g. periodized log sing : $g(s) = \ln\left(4\sin^2\frac{s}{2}\right)$ has $g_n = \begin{cases} 0 & n=0 \\ -\frac{1}{|n|} & \text{otherwise.}\end{cases}$ [LIE, Thm 8.21, algebra].



note : $g'(s) = \cot s/2$ helps proof.

note since $\sum_{n\in\mathbb{Z}} |g_n|^2 < \infty$, $g \in L^2(0,2\pi)$.

Then $w_j = \frac{2\pi}{N}\Big[\sum_{n=0}^{N/2-1} \underbrace{g_n e^{\overset{is_jn}{\frac{2\pi ijn}{N}}} + \overset{=\overline{g_n},\text{ since } g \text{ real.}}{g_{-n}} e^{-\frac{2\pi ijn}{N}}}_{2\,\mathrm{Re}(g_n e^{\frac{2\pi ijn}{N}})} + \underbrace{\tfrac{1}{2}(g_{N/2}e^{\pi ij} + g_{-N/2}e^{-\pi ij})}_{(-1)^j \mathrm{Re}\, g_{N/2}}\Big]$

True for any real g.

For our g_n,

$$W_j = \frac{2\pi}{N}\left[-\sum_{n=1}^{N/2-1}\frac{2}{n}\cos ns_j - (-1)^j\frac{1}{N}\right] \qquad \text{done.}$$

Note: the $\approx$ in above are exact for $f \in \underbrace{\mathrm{Span}\{e^{ins}\}_{n=-N/2}^{N/2}}_{T_N \text{ 'trigonometric polynomials'}}$, can check.

Now can split kernel of $D - \eta S$ into $\underset{\text{usual PTR}}{\text{analytic}(s,t)} + \underbrace{\log\left(4\sin^2\frac{s-t}{2}\right)}_{\text{new weights, shifted cyclically.}} \cdot \text{analytic}(s,t)$

eg S has kernel (wrt. parameter $0 \le s < 2\pi$):

$$\frac{i}{4}H_0(k|y(t)-y(s)|)\underbrace{|y'(s)|}_{\text{speed.}} = \underbrace{\frac{-1}{4\pi}J_0(k|y(t)-y(s)|)\,|y'(s)|}_{M_1(t,s)\text{ anal.}} \cdot \ln\left(4\sin^2\frac{t-s}{2}\right) + \underbrace{M_2(t,s)}_{\text{analytic, since the log part precisely removed!}}$$

$M_2(t,s)$ defined by this, apart from $M(s,s)$, for which exists formula.

Similar for D... see Kress '91, or Colton-Kress '98.

Scheme: Nyström matrix $A_{ij} = \underset{\text{PTR weights (fixed)}}{\frac{2\pi}{N}} M_2(s_i,s_j) + \underbrace{W_{|i-j|}}_{\text{why? this shifts singularity to be at } j=i \text{ for row } i.} M_1(s_i,s_j)$

'Circulant' matrix: each row is previous cyclically shifted 1 to right.

Note: incompatible w/ FMM since weights dep. on j and i so can't be treated as fixed charges.

Other objects you should see:

A) Sobolev spaces (type of Hilbert space)

recall $L^2[0,2\pi) := \{\text{functions } f : \underbrace{\int_0^{2\pi}|f(x)|^2dx}_{\|f\|^2_{L^2} = 2\pi\sum_{n\in\mathbb{Z}}|f_n|^2} < \infty\}$, loosely.

Defn: (Sobolev space order s)

$H^s[0,2\pi) := \{\text{funcs } f : \underbrace{\sum_{n\in\mathbb{Z}}(1+n^2)^s|f_n|^2}_{\|f\|^2_{H^s}} < \infty\}$ v. common in PDE analysis.

$H^0 = L^2$, $s>0$ enforces faster decay of Fourier coeffs. $\Rightarrow$ smoother than L^2.

Eg $g(x) = \ln(4\sin^2\frac{x}{2}) \in H^s[0,2\pi] \quad \forall s < 1/2$ since $\sum(1+n^2)^s\frac{1}{n^2} < \infty$.

Thm: Let $s > 1/2$, $f \in H^s$, then $f \in C[0,2\pi]$. ← periodic

Pf: for each x, $\left(\sum_{n\in\mathbb{Z}}|f_n e^{inx}|\right)^2 \overset{C.S.}{\le} \overbrace{\left(\sum_{n\in\mathbb{Z}}\frac{1}{(1+n^2)^s}\right)}^{\text{converges for } s>1/2}\sum_{n\in\mathbb{Z}}(1+n^2)^s|f_n|^2$

So Fourier series abs. conv, f uniform lim. of cont. funcs.

Thm: Let $f \in H^s$, then $\frac{df}{dx} \in H^{s-1}$ ← derivative is one order less smooth.

pf: fourier coeffs of f' are $i n f_n$. □

So, $H^1(a,b) = \{ f : \underbrace{\int_a^b |f(x)|^2 dx + \int_a^b |f'(x)|^2 dx}_{\text{since } 2\pi \sum (1+n^2)|f_n|^2} < \infty \}$

$H^1(\mathbb{R}^d)$ is higher-dim analog, common for PDE.

Thm: single-layer op. S is bnded from H^s to H^{s+1}, ie S behaves like "1 order of integration." (smoothing, makes coeffs decay more)

pf sketch: i) singularity of S is $g(s-t) = \ln(4 \sin^2 \frac{s-t}{2})$ ↳ convolution kernel.
which has $|g_n| \sim \frac{1}{|n|}$

ii) Applying convolution op. $h(t) = \int g(t-s) f(s) ds = (g * f)(t)$, is $h_n = f_n g_n$ in Fourier space. (check it!).

Thm: $D : H^s \to H^s$ bnded (order 0)
$T : H^s \to H^{s-1}$ bnded, ie like derivative of 1 order.

see [LIE Ch. 8], [CK] books

Suggests that $\underset{\text{order } +1}{T}\underset{\text{order } -1}{S}$ is order 0. True: $TS = -I/4 + (D^*)^2$

called 'Calderón identity'; numerically can use to 'precondition' nasty T into Id+cpt.

B) Calderón projectors (Helmholtz case) Recall a "projection op." P obeys $P^2 = P$ as operators.

Recall interior GRF $\quad -Du^- + Su_n^- = \begin{cases} u & \text{in } \Omega \\ 0 & \text{in } \mathbb{R}^d \backslash \bar{\Omega} \end{cases}$

taking $x \to \partial\Omega^-$, & using JR1-4, $\begin{cases} \text{values:} & -(D - 1/2)u^- + Su_n^- = u^- \\ \text{n-derivs:} & -Tu^- + (D^* + 1/2)u_n^- = u_n^- \end{cases}$

ie $\left(\underbrace{-\begin{bmatrix} D & -S \\ T & -D^* \end{bmatrix}}_{H} + \begin{bmatrix} I/2 & \\ & I/2 \end{bmatrix} \right) \begin{bmatrix} u^- \\ u_n^- \end{bmatrix} = \begin{bmatrix} u^- \\ u_n^- \end{bmatrix}$ $\quad = \frac{1}{2}\text{Id}$.

so $P_- := (1/2 - H)$ is identity in lin. subspace V_- of interior bdry data pairs.

Instead taking $x \to \partial\Omega^+$ gives opposite jumps, ⇒ $\underbrace{(1/2 + H)}_{=:P_+} \begin{bmatrix} u^- \\ u_n^- \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

But haven't yet shown P_- is a projection!

Showing P_- is actually a projection : ④ 3/1/12

$\forall \tau, \sigma$, know $\mathcal{D}\tau + \mathcal{S}\sigma$ is an interior Helmholtz solution, say u, in which case $\begin{bmatrix} u^- \\ u_n^- \end{bmatrix} \in V_-$

$\Rightarrow$ Using JRs as before, $P_-\begin{bmatrix}-\tau \\ \sigma\end{bmatrix} = \begin{bmatrix}u^- \\ u_n^-\end{bmatrix}$

$$\text{so } P_-^2\begin{bmatrix}-\tau \\ \sigma\end{bmatrix} = P_-\left(P_-\begin{bmatrix}-\tau \\ \sigma\end{bmatrix}\right) = P_-\begin{bmatrix}u^- \\ u_n^-\end{bmatrix} \overset{\text{since } P_- \text{ acts as Id in } V_-}{=} \begin{bmatrix}u^- \\ u_n^-\end{bmatrix} = P_-\begin{bmatrix}-\tau \\ \sigma\end{bmatrix}$$

True $\forall \tau, \sigma$, so $P_-^2 = P_-$ as ops. □

Since $P_- = \frac{1}{2} - H$, $(\frac{1}{2} - H)^2 = \frac{1}{4} - \cancel{H} + H^2 = \frac{1}{2} - \cancel{H}$ so $H^2 = \frac{1}{4}$, ie $\begin{bmatrix} D^2 - ST & -DS + SD^* \\ TD - D^*T & -TS + D^{*2}\end{bmatrix} = \begin{bmatrix}1/4 & 0 \\ 0 & 1/4\end{bmatrix}$

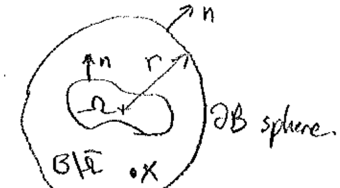
Calderón identities!

Thm (exterior GRF) : let $(\Delta + k^2)u = 0$ in $\mathbb{R}^d \setminus \bar\Omega$ & u sat. radiation condition @ ∞, then

$$-\mathcal{D}u^+ + \mathcal{S}u_n^+ = \begin{cases} 0 & \text{in } \Omega \\ -u & \text{in } \mathbb{R}^d\setminus\bar\Omega \end{cases}$$

[CK book Thm. 2.4]

Proof:



Let B be ball centered at 0 enclosing Ω, radius r

i) We first show $\int_{\partial B} |u|^2 ds = O(1)$ as $r \to \infty$

we have the identity, by expanding,

$$\int_{\partial B} \left|\frac{\partial u}{\partial r} - iku\right|^2 ds = \int_{\partial B} \left|\frac{\partial u}{\partial r}\right|^2 + k^2|u|^2 + 2k\,\overbrace{\mathrm{Im}\, u\frac{\partial \bar u}{\partial r}}^{-\text{flux leaving sphere}}\,ds \qquad (\dagger)$$

Also, in any region R in which u a Helmholtz soln, we have "flux balance" (FB):

$$\mathrm{Im}\int_{\partial R} u\bar u_n\, ds = \mathrm{Im}\int_R u\underbrace{\Delta\bar u}_{-k^2\bar u} + \nabla u\cdot\nabla\bar u\, dx \quad \text{by G1I.}$$

$$= 0 \quad \text{since RHS purely real.}$$

interpret as net flux entering R.

Apply FB to $R = B\setminus\bar\Omega$ gives

$$2k\,\mathrm{Im}\int_{\partial B} u\frac{\partial\bar u}{\partial r}\, ds = \underbrace{2k\,\mathrm{Im}\int_{\partial\Omega} u\bar u_n\, ds}_{\text{a bounded number } F,\text{ dep on } u.}$$

Combine w/ $(\dagger)$ gives

$$\lim_{r\to\infty}\int_{\partial B}\left|\frac{\partial u}{\partial r}\right|^2 + k^2|u|^2 ds = -F + \underbrace{\lim_{r\to\infty}\int_{\partial B}\left|\frac{\partial u}{\partial r} - iku\right|^2 ds}_{=0 \text{ by rad. cond.}}$$

LHS is sum of nonneg. terms, so each bounded, $O(1)$

ii) Now use this to show surface term in GRF on ∂B vanishes as $r \to \infty$. Let $x \in B\setminus\bar\Omega$,

$$\int_{\partial B}\left[u(y)\frac{\partial\Phi(x,y)}{\partial n_y} - u_n(y)\Phi(x,y)\right] ds_y = \underbrace{\int_{\partial B} u\left[\frac{\partial\Phi}{\partial n_y} - ik\Phi\right] ds_y}_{=:I_1} - \underbrace{\int_{\partial B}\Phi(u_n - iku)\, ds_y}_{=:I_2}$$

claim $I_1, I_2 \to 0$ as $r \to \infty$:

⑤ 3/1/12

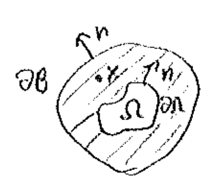
$$\frac{\partial\Phi(x,y)}{\partial n_y} - ik\Phi(x,y) = o\left(\frac{1}{r^{\frac{d-1}{2}}}\right) \quad \text{since } \Phi(x,\cdot) \text{ radiating soln.}$$

by C-S. $I_1^2 \le \underbrace{\int_{\partial B} |u|^2 ds}_{O(1)} \cdot \int_{\partial B} \underbrace{\left|\frac{\partial\Phi(x,y)}{\partial n_y} - ik\Phi(x,y)\right|^2}_{o\left(\frac{1}{r^{d-1}}\right)} ds_y = o(1)$ as $r\to\infty$.

surf. area is $O(r^{d-1})$

for I_2, $\Phi(x,\cdot) = O\left(\frac{1}{r^{\frac{d-1}{2}}}\right)$ & u radiating, so $I_2 \to 0$ as $r\to\infty$.

iii) We apply interior GRF to $B\setminus\bar{\Omega}$ gives,



$$-\int_{\partial\Omega+\partial B} u_n(y)\Phi(x,y) - u(y)\frac{\partial\Phi}{\partial n_y}(x,y)\, ds_y = \begin{cases} u(x) & x\in B\setminus\bar{\Omega} \\ 0 & x\in\Omega \end{cases}$$

since $\partial\Omega$ normal points into $B\setminus\bar{\Omega}$

in ii) we showed this term vanishes as $r\to\infty$

True for each r. Finally take $\lim r\to\infty$. QED.

May now finish Calderón Projectors:

apply exterior GRF, take $x\to\partial\Omega^+$ & use JR's gives, $P_+\begin{bmatrix}u^+\\u_n^+\end{bmatrix} = \begin{bmatrix}u^+\\u_n^+\end{bmatrix}$, $P_-\begin{bmatrix}u^+\\u_n^+\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$
(for u any radiative Helm. soln. in $\mathbb{R}^d\setminus\bar{\Omega}$)

holds for all $\begin{bmatrix}u^+\\u_n^+\end{bmatrix}\in V_+$

by identical proof as P_-, we then get P_+ is a projection.

Lemma: $V_+ \oplus V_- = \begin{bmatrix}L^2(\partial\Omega)\\L^2(\partial\Omega)\end{bmatrix}$ Pf: $I = \tfrac12 + H + \tfrac12 - H = P_+ + P_-$

so, $\forall \sigma,\tau$, $\begin{bmatrix}-\tau\\\sigma\end{bmatrix} = P_+\begin{bmatrix}-\tau\\\sigma\end{bmatrix} + P_-\begin{bmatrix}-\tau\\\sigma\end{bmatrix}$, is a decomposition into V_+ & V_-. □

Summary: $P_+V_+ = V_+$, $P_+V_- = \{0\}$ and $P_+P_- = P_-P_+ = 0$
$P_-V_+ = \{0\}$, $P_-V_- = V_-$

Thus P_+, P_- are complementary projectors.

Notes:
- Shipman-Venakides paper, eg. 2003, have clear explanation of this.
- contrary to statement of Kress in his acoustic notes, $2H$ is not a projection. (it is a reflection, since $(2H)^2 = I$)
- We haven't shown $V_+ \perp V_-$, ie that projections are orthogonal. This would require $P_+ = P_+^*$, etc; I don't believe holds.

FIN