[lec 4. cont]   ④ 1/12/12

Is $f(x) = x \ominus 1$ bkw st? $\quad \tilde{f}(x) = [x(1+\varepsilon_1) - 1](1+\varepsilon_2) \overset{\text{set}}{=} x(1+\varepsilon_3) - 1 =: f(\tilde{x})$.

how big is ε_3? $\quad x\varepsilon_3 = x(\varepsilon_1+\varepsilon_2+O(\varepsilon^2)) - \varepsilon_2$

so $\quad \varepsilon_3 = \varepsilon_1+\varepsilon_2 - \frac{\varepsilon_2}{x} = O(\varepsilon_{mach}) + \frac{1}{x}O(\varepsilon_{mach})$

So as $x \to 0$, not bkw stable. But, is stable.

Some algs unst! (eg. poly roots)

meaning: computed soln $\tilde{y}$ sat. $(A+\delta A)\tilde{y} = b$ exactly for some δA with $\|\delta A\|/\|A\| = O(\varepsilon_{mach})$

Take-home msg: algs (in Matlab, LAPACK, etc) for solving $Ay = b$ are bkw stable (A is data 'x', y is answer 'f(x)')

$m=n$ (nonsing. square): QR is (Thm 16.2 NLA)

Gauss elim. w/ partial pivoting is Ch. 22. (if avoid incredibly rare pathological matrices)

$m>n$: least-squares soln, ie find x st. $\|Ax-b\|$ minimized.

is bkw stable via SVD (Thm 19.4)

How do? $A\backslash b$ mldivide does it.

or, explicitly, $A = U\Sigma V^*$ so $x = \underbrace{V^*\Sigma^{-1}U}_{\text{pseudoinverse}} b =: A^+ b$

Reminder: $\frac{\|\tilde{y}-y\|}{\|y\|}$ may not be small, ie y itself inaccurate! What is only cause of this in bkw.-stable alg? κ v. large.

But in this case, such is bkw st: as good as one could hope for!

How accurate is $\tilde{y}$, ie $\tilde{f}(x)$? For any bkw-stable alg: $x \to f(x)$

Thm (15.1): if cond # is $\kappa(x)$ for problem $f(x)$, alg is bkw st, and computer obeys floating pt axioms, then rel. err. satisfies

$$\frac{\|\tilde{f}(x) - f(x)\|}{\|f(x)\|} = O(\kappa(x)\,\varepsilon_{mach})$$

- Ie, error is κ times worse. If $\kappa > 10^{16}$ you lose all digits of $f(x)$, or of y. But it still holds that $(A + O(\varepsilon_{mach}))\tilde{y} = b$ exactly!
- Pf easy: by defn. $\tilde{f}(x) = f(\tilde{x})$ (a) for some $\frac{\|\tilde{x}-x\|}{\|x\|} = O(\varepsilon_{mach})$ (b)

Defn. of κ: $\frac{\|f(\tilde{x}) - f(x)\|}{\|f(x)\|} \le (\kappa + o(1))\frac{\|\tilde{x}-x\|}{\|x\|}$ & sub in (a) & (b). QED.

since not infinitesimal.

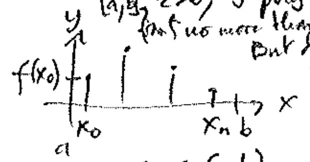
Stability & rounding done.

Lec 4 part 2. ① 1/17/12

Interpolation [from Kress NA §8.1]

Approx func f on $[a,b]$ by degree-n poly $p(x) = \sum_{k=0}^{n} a_k x^k$ ← monomials ← why good idea? Weierstrass: for each $[a,b]$, $\varepsilon>0$, $\exists$ poly differing from f no more than ε. But degree unknown (finite?).

If choose $n+1$ distinct pts ('nodes') to make f & p match, we've already solved this:



$p(x_j) = y_j \quad j=0,\dots n.$ $\quad x_j \in [a,b]$.

so
$$\underbrace{\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots \\ 1 & x_1 & x_1^2 & \\ \vdots & \vdots & & \\ 1 & x_n & & \end{bmatrix}}_{M} \begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} f(x_0) \\ \vdots \\ f(x_n) \end{bmatrix}$$
proved $\det M \neq 0$ in lec 1 $\Rightarrow$ soln exists, unique.

Let $\ell_k(x) = \prod_{\substack{j=0 \\ j\neq k}}^{n} \frac{x-x_j}{x_k-x_j} \quad k=0,\dots n$ called Lagrange basis (1794)

Prop: unique interp. poly $p = \sum_{k=0}^{n} y_k \ell_k$ why? pf: $\ell_k(x_i) = \begin{cases} \prod_{j\neq k} \frac{x_k-x_j}{x_k-x_j} = 1, & i=k \\ 0 & i\neq k \text{ since one factor is } x_i - x_i \end{cases} = \delta_{ik}$

so $p(x_i) = \sum y_k \delta_{ki} = y_i \quad \forall i$ is a solution & unique.

Note:
- n large (>30) may cause stability probs since $\sup_{x\in[a,b]} |\ell_k(x)|$ exp. large
- Newton 1676 realised more practical method 'divided differences' we don't need.
- map $f \to$ its unique interp poly p through $\{x_j\}$ is linear: $p = L_n f$ → $L_n : C[a,b] \to \mathbb{P}_n$ ← space of degree-n poly's.

  for any $p \in \mathbb{P}_n$, $L_n p = p$ so what kind of op. is L_n? projection: $L_n^2 = L_n$.

Error of interp. $L_n f - f$ is a func.

recall $C^k[a,b]$ space of k-times cont. diff'ble funcs, ie k^{th} deriv is cont.

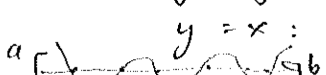
Thm (8.10) Let $f \in C^{n+1}[a,b]$, then for each $x\in[a,b]$ there exists $\xi \in [a,b]$ st.
$$f(x) - L_n f(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{j=0}^{n} (x-x_j)$$

- So if you know $|f^{(n+1)}(x)| \le C$ in $x\in[a,b]$ you get an error estimate ← in math 'estimate' means rigorous bound!

pf: trivial if $x = x_j$

Fix $x \neq x_j$ & define $g(y) := f(y) - L_n f(y) - \prod_{j=0}^{n}(y-x_j) \frac{f(x) - L_n f(x)}{\prod_{j=0}^{n}(x-x_j)} \quad y\in[a,b]$

set $y = x_j$: $\underbrace{f(y) - L_n f(y)}_{0}$, $\underbrace{\prod_{j=0}^{n}(y-x_j)}_{0}$ $\Rightarrow g(x_j) = 0 \quad j=0,\dots n$



$y = x$: $g(x) = 0$ too! (that was why constructed), so g has $n+2$ zeros in $[a,b]$

By Rolle's thm. g' has $\ge n+1$ zeros.

etc. ... $g^{(n+1)}$ has ≥ 1 zero, call it ξ

Set $y = \xi$ & eval $g^{(n+1)}(\xi)$: $0 = f^{(n+1)}(\xi) - 0 - (n+1)! \frac{f(x) - L_n f(x)}{\prod_{j=0}^{n}(x-x_j)}$ ← since degree n QED sneaky!