Lec 6. M126 · guess n=2 ws. ① 1/24/12

HW2. for log plots, careful to use semilogy, loglog, etc, rather than take log of data. is ln not $\log_{10}$.
↓
demo importance of Chebychev vs equally-spaced nodes!

recall Quadr $Q_n(f) = \sum_{j=0}^{n} w_j f(x_j)$ $\exists$ unique $\{w_j\}$ st. Q_n exact $\forall f \in \mathbb{P}_n$.

last time: Interpolatory quad on $[a, b]$, $n=1$, ie $n+1 = 2$ nodes, choose $x_0 = a$, $x_1 = b$, get $Q_1(f) = h\frac{f(a)+f(b)}{2}$ trapezoid rule.

Thm (9.4): Let $f \in C^2[a,b]$, then $\left|\int_a^b f(x)dx - Q_1(f)\right| \le \frac{1}{12}\|f''\|_\infty h^3$

Pf [Thm 12.1 LIE]: Peano Kernel $K(x) = \frac{1}{2}(x-a)(b-x) \geq 0$ on $[a,b]$, $K'' = -1$ on (a,b). $K' = \frac{b-a}{2}$, $K' = -\frac{b-a}{2}$

$\int_a^b K(x) f''(x)dx = -\int K' f' dx + \underbrace{[Kf']_a^b}_{0} = \int K'' f\, dx - [K'f]_a^b = -\int_a^b f(x)dx + \frac{h}{2}(f(a)+f(b))$

magnitude $\le \underbrace{\int_a^b K(x)\,dx}_{\frac{h^3}{6}} \cdot \|f''\|_\infty$ QED.

May split interval into smaller & apply above to each: a h h ... h b "composite trapezoid rule"

error $\le \frac{1}{12}\|f''\|_\infty h^3 \cdot \underbrace{\frac{b-a}{h}}_{\#\text{ intervals}}$ $Q_{ctr}(f) = h\left(\frac{1}{2}f(a) + f(a+h) + \cdots + f(b-h) + \frac{1}{2}f(b)\right)$

$= \frac{b-a}{12}\|f''\|_\infty h^2 = O(h^2)$ convergence algebraic, order = # of nodes (n+1)
← low-order.

Can increase n: more points on single interval, eg $n=2$ Simpsons' (1743)

Say choose w/ equispaced pts. a ├─┤ b ← single interval. h

Guesses for w_j as $n \to \infty$? w_j are $\int_a^b \ell_j(x)\,dx$
$\Rightarrow$ exp. large & oscillatory → bad for roundoff.

Another way in which negative weights bad: Convergence.
(§9.2)

Consider seq. $(Q_n)_{n=0}^{\infty}$ of schemes $Q_n(f) := \sum_{j=0}^{n} w_j^{(n)} f(x_j^{(n)})$

Defn (Q_n) conv. if $Q_n(f) \to Q(f)$ as $n \to \infty$, $\forall f \in C[a,b]$ nice property! (but could be slow). (Recall impossible for interp! L_∞ error)

• Thm (Szegö) (Q_n) conv. $\iff$ (Q_n) conv. for all poly's & $\exists C$ st. $\sum_{j=0}^{n} |w_j^{(n)}| \le C$, $\forall n$

note: these means if weights blow up as $n \to \infty$, cannot be conv! (eg. equispaced)

Facts 1) $\mathbb{P}$ = poly's 'dense' in $C[a,b]$, meaning: $\forall f \in C[a,b]$ & $\forall \varepsilon > 0$, $\exists p \in \mathbb{P}$ st. $\|f-p\|_\infty \le \varepsilon$
↑ if need intuition: like $\mathbb{Q} \subset \mathbb{R}$, but metric space is $C[a,b]$ w/ sup norm. Weierstrass.

2) each Q_n is lin. op: $C[a,b] \to \mathbb{R}$ w/ $|Q_n(f)| \le \|f\|_\infty \sum_{j=0}^{n} |w_j^{(n)}| \le C\|f\|_\infty$.
so in op. norm, this is $\|Q_n\|_\infty$ operator norm.

Pf: use facts 1 & 2 w/ Banach-Steinhaus thm:
let (Q_n) be seq. of bnd lin. ops, & Q bnd lin. op, $X = C[a,b]$ Banach space.

pointwise convergence $\iff$ (Q_n) uniformly bnded & convergent on some dense subset
$\forall f \in X, \lim_{n\to\infty} \|Q_n f - Qf\| = 0$ $\|Q_n\| \le C$, $\forall n$ $\exists U \subset X$ dense st. $\forall f \in U$, $\lim_{n\to\infty} \|Q_n f - Qf\| = 0$.

Banach–Steinhaus is a variant of "Principle of Uniform Boundedness"? both std. in func. anal. ② 1/24/12
This is pretty abstract, ← the $\boxed{\Rightarrow}$ in B-S.
so let's prove the non-P.U.B. part, the $\boxed{\Leftarrow}$ in thm: [ptwise conv. $\Leftarrow$ dense conv. & unif bnd]

For any $\varepsilon > 0$,

$$Q_n f - Qf = Q_n f - Q_n p + Q_n p - Qp + Qp - Qf$$

can find $p \in \mathbb{P}$ ("our 'U'") s.t. $\|p - f\|_\infty \le \varepsilon$

Take abs val & use tri ineq:

$$|Q_n f - Qf| \le \underbrace{|Q_n f - Q_n p|}_{\substack{\le C\|f-p\|_\infty \\ \le C\varepsilon}} + \underbrace{|Q_n p - Qp|}_{\substack{\le \varepsilon \\ \forall n > N \text{ for some } N \text{ suff. large.}}} + \underbrace{|Qp - Qf|}_{\substack{\le (b-a)\varepsilon \\ \text{if you like, } \|Q\|}}$$

$$\le \underbrace{(C + 1 + b - a)}_{\text{fixed const.}}\varepsilon$$

So for each $\delta > 0$, choose $\varepsilon = \frac{\delta}{C+1+b-a}$ & $\exists N$ st. $\forall n > N$, $|Q_n f - Qf| < \delta$. QED.

"up & over"
$$\begin{array}{ccc} Q_n p & \longrightarrow & Qp \\ \uparrow & & \downarrow \\ Q_n f & \cdots\cdots & Qf \end{array}$$
bound by $\le$ sum of 3 dists.

· Also called "$\varepsilon/3$" argument, common in func. anal.

Positive weights is enough:

Corollary (9.11, Steklov): if (Q_n) conv for all polys, & $w_j^{(n)} \geq 0$, then (Q_n) conv.

pf: $\|Q_n\|_\infty = \sum_{j=0}^{n} |w_j^{(n)}| \underset{\text{nonneg.}}{=} \sum_{j=0}^{n} w_j^{(n)} = Q_n(1) \xrightarrow[\text{1 is a poly}]{} Q(1) = \int_a^b 1\,dx = b-a$

so $\exists C$ st. $\|Q_n\|_\infty \le C$ $\forall n$, use Szegö.

∘ Also minimal round-off error since sizes of weights as small as poss.

∘ Eg: composite trap. conv. (+ve w's, conv. $\forall$ poly's since each has $\|p''\|_\infty < \infty$). , But equispaced

Now improved quadr. schemes on $[a,b]$ … w/ positive weights…

Gaussian Quadrature (§9.3)

: optimal choice of nodes → queen of quadratures. on $[a,b]$

→ vs. straight in.

↳ get $n=2$ degree 5 exact, generally can do degree $2n+1$ exact, compared to only n for Newton-Cotes w/ general nodes.
↳ defines a Gaussian quad' w/ $n+1$ nodes.

Let's show why:

∘ orthogonality for funcs. $f \perp g \iff 0 = (f,g) := \int_a^b f(x)g(x)\,dx$

Lemma (9.13) Let $x_0, \dots x_n$ be distinct nodes of a Gaussian quadr.

Then $q_{n+1}(x) := \prod_{j=0}^{n} (x - x_j) \perp p \quad \forall p \in \mathbb{P}_n$

q_{n+1}:

pf: $q_{n+1}p \in \mathbb{P}_{2n+1}$ so $\int_a^b q_{n+1}(x)p(x)\,dx \overset{\text{by Gauss.}}{=} \sum_{k=0}^{n} w_k \underbrace{q_{n+1}(x_k)}_{\text{vanish at nodes.}} p(x_k) = 0$ QED.