Lec. 7. · projector · ① 1/26/12

HW2 debrief:

finish Gauss n=2 WS. → $2\beta\alpha^2 = \int_{-1}^{1} x^2 dx = 2/3$, $2\beta\alpha^4 = \int_{-1}^{1} x^4 dx = 2/5$ $\alpha = \sqrt{3/5}$, $\beta = 5/9$, $w_1 = 2 - 2\beta = 8/9$

Do §9.3 from ②-③ 1/24/12. $\mathbb{P}_5$ exact.

Claim $2n+1$ is highest poss. degree for $(n+1)$-node quadr.

Pf: $p = \prod_{j=0}^{n} (x - x_j)^2 \in \mathbb{P}_{2n+2}$ has $Q_n(p) = 0$ but $Q(p) > 0$.

Thm: (9.18) Gauss weights non-negative. Pf $l_k(x_j) = \delta_{jk}$ so $l_k^2(x_j) = \delta_{jk}$ so

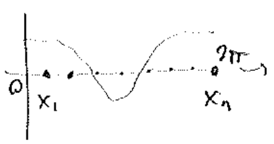
$$\forall k: \quad \underset{\text{nonneg.}}{0 <} \int_a^b l_k^2(x)\,dx \underset{\substack{\text{exact} \\ \text{since } l_k^2 \in \mathbb{P}_{2n}}}{=} \sum_{j=0}^{n} w_j l_k^2(x_j) = w_k.$$

Cor: Gaussian quadr. convergent (last time).

There are error bnds for Gauss quadr, eg order-n rule err $\leq \frac{\|f^{(2n+2)}\|_\infty}{(2n+2)!} \cdot \overbrace{\int_a^b q_{n+1}^2(x)\,dx}^{\text{gets exp. small as } n\to\infty}$.

May generalize to weighted quadr. $Q(f) = \int_{-1}^{1} f(x) W(x)\,dx$, has some uses.

Periodic Quadrature §9.4.



$f(x+2\pi) = f(x)\ \forall x.$ $Q(f) = \int_0^{2\pi} f(x)\,dx$

$$Q_n(f) = \underbrace{\frac{2\pi}{n}}_{w_j \text{ equal.}} \sum_{j=1}^{n} f\Big(\underbrace{\frac{2\pi j}{n}}_{\text{equally spaced nodes.}}\Big)$$

Thm (9.27) Let $f \in C^{2m+1}(\mathbb{R})$ be 2π-periodic, $m \geq 1$, then $\underbrace{|Q_n(f) - Q(f)|}_{\text{error}} \leq \underset{\substack{\text{const indep.} \\ \text{of } f.}}{c_m} \int_0^{2\pi} |f^{(2m+1)}(x)|\,dx \cdot \frac{1}{n^{2m+1}}$

Pf: see [NA], Euler-Maclaurin.

Ie, smoother f gives higher-order algebraic convergence.

If $f \in C^\infty$, then error $= O(n^{-m})$ for each $m \geq 0$, called "super-algebraic" convergence.

But if f analytic, do even better: exponential conv.

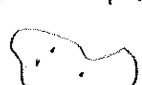
→ [2011o.pdf slides] write Davis thm. then ↳ First review complex: $f(z)$ holomorphic in $D \subset \mathbb{C}$: means analytic at each pt. in D.

Eg. $\frac{1}{1+25x^2}$ holomorphic in $\mathbb{C} \setminus \{i/5, -i/5\}$. ← simple poles.

simple pole @ $z=a$: $f(z) = \frac{b}{z-a}$ ← residue.

generally at simple pole, $f(z) = \frac{b}{z-a} + \underbrace{c_0 + c_1 z + \cdots}_{\text{Taylor.}}$

Residue thm: if f holomorphic in D apart from finite # poles,



$$\underset{\text{line integral}}{\int_{\partial D} f(z)\,dz} = 2\pi i \sum_{\text{poles}} (\text{residue of each pole}).$$

Pf of thm (slides).

· May also derive from trigonometric interpolation, ie Fourier series truncated at terms $\pm n/2$, is also exp. accurate.

[Lec 7 middle of] & choose $\{w_j\}$ as usual, ③ 1/24/12

Converse of this holds: Lemma 9.14: if $\{x_j\}$ nodes sat. $q_{n+1} \perp \mathbb{P}_n$ γ, it's a Gauss. quad.

Pf: recall interpolatory quad. has $\sum w_k f(x_k) = \int_a^b (L_n f)(x)\,dx \quad \forall f \in C[a,b]$

claim each $p \in \mathbb{P}_{2n+1}$ can be written $p = L_n p + q_{n+1} q$ for some $q \in \mathbb{P}_n$

why? $p - L_n p = 0$ at $\{x_j\}$, so q can have at most $(2n+1) - (n+1) = n$ zeros.

So $\int p(x)\,dx = \int (L_n p)(x)\,dx + \underbrace{\int q_{n+1}(x)\, q(x)\,dx}_{\text{chosen orthog.}} \underset{\substack{\text{since} \\ \text{interp.} \\ (\text{degree } n)}}{=} \sum w_k p(x_k)$ QED.

So, if can find q_{n+1}, a degree-$(n+1)$ poly, orthog to $\mathbb{P}_n$, with all roots in $[a,b]$,

neat: the <u>roots</u> give the <u>nodes</u>!

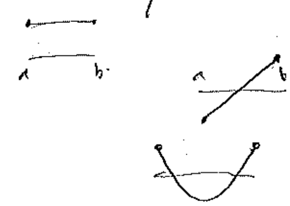
ORTHOGONAL POLYNOMIALS (useful anyway):

Lemma 9.15: $\exists$ unique seq. $(q_n)_0^\infty$ w/ $q_0 = 1$ & $q_n(x) = x^n + p(x)$, $p \in \mathbb{P}_n$

which are <u>mutually</u> orthog. ie $(q_n, q_m) = 0 \quad \forall m < n$, and $\operatorname{span}\{q_0, \dots, q_n\} = \mathbb{P}_n$.

Pf: $1, x, x^2, \dots$ are lin. indep. on $[a,b]$, so <u>Gram-Schmidt</u> unique:

$$q_0 = 1$$
$$q_1 = x - \frac{(x, q_0)}{(q_0, q_0)} q_0$$
$$q_2 = x^2 - \frac{(x^2, q_1)}{(q_1, q_1)} q_1 - \frac{(x^2, q_0)}{(q_0, q_0)} q_0$$
$$\vdots$$
$$q_n = x^n - \sum_{j=0}^{n-1} \frac{(x^n, q_j)}{(q_j, q_j)} q_j$$



& these $n+1$ L.I. elements of $\mathbb{P}_n$ (an $n+1$-dim vector space) must span it.

in HW3 you'll prove this can be done via 3-term recurrence, ie q_{n+1} involves q_n & q_{n-1} only.

- 'Legendre' poly's (but above not std normalization): unique seq. of orthog poly's on $[-1,1]$ w/ <u>unweighted</u> inner product (f,g).

Lemma 9.16 q_n has n simple zeros all in $[a,b]$ (... good, so they give a Gauss quad.!)

pf. $\forall n \geq 1$, $q_n \perp q_0$ ie $\int q_n = 0$ so q_n has ≥ 1 zeros $x_1 \cdots x_m$ in $[a,b]$

supp. $m < n$, then $r_m := \prod_{j=1}^{m} (x - x_j) \in \mathbb{P}_{n-1}$ so is $\perp q_n$

but $\int r_m q_n \neq 0$ since $r_m q_n$ has fixed sign, not $\equiv 0$. $\Big\}$ contradiction. $\Rightarrow m = n$.

In practice, how compute $\{x_j\}_{j=0}^{n}$? They are eigvals of

$$\begin{pmatrix} 0 & \beta_1 & & & \\ \beta_1 & 0 & \beta_2 & & \\ & \beta_2 & 0 & \beta_3 & \\ & & \beta_3 & \ddots & \ddots \\ & & & \ddots & \end{pmatrix}$$

tridiagonal matrix $\beta_n := \frac{1}{2\sqrt{1-(2n)^{-2}}}$

& $\{w_j\}$ come from eigenvectors. "Golub-Welsh"

See code gauss.m

This is $O(n^3)$ slow! ($n < 10^2$ ok, otherwise slow)

Glaser-Liu-Rokhlin 2007 has better idea which is $O(n)$: find x_{j+1} from x_j by Taylor expansion, coeffs given by Legendre poly recurrence.