Lec. 8 · [M126] · copy 12 or page of Ch 12 NA. ① 1/31/12.

HW3 debrief:

Integral Eqns: §12 (NA)

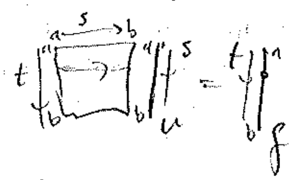
given interval $[a,b]$, func f on $[a,b]$, kernel func k on $[a,b]^2$

solve $\underbrace{\int_a^b k(t,s)\,u(s)\,ds}_{(Ku)(t)} = f(t) \quad \forall t \in [a,b]$ ← right hand side.

Fredholm 1st kind.

or 2nd kind $\quad u(t) + (Ku)(t) = f(t) \quad \forall t\in[a,b]$

functional eqn: $Ku = f$, visualize like $A\vec{x} = \vec{b}$, ie



← ie $f(t)$ = inner prod of $k(t,\cdot)$ & u

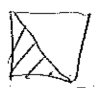
• What is $(K^2u)(t)$? $= \int k(t,s)\underbrace{(Ku)(s)}_{\text{write out}}\,ds = \int k(t,s)\int k(s,r)u(r)\,dr\,ds$

$= \int k'(t,r)\,u(r)\,dr$

where k' is kernel of K^2 so $k'(t,r) = \int k(t,s)\,k(s,r)\,ds$

[like matrix prod. $(AB)_{ik} = \sum_j a_{ij} b_{jk}$]

• if $k(t,s) = 0$ for $s > t$



diagonal. $t=s$

'lower-triangular', they're called Volterra, not Fredholm,

can be written $\int_a^{t} k(t,s)\,u(s)\,ds = f(t)$ (note: t)

has unique soln, / won't concern us.

eg $k\equiv 1$: $\int_0^t u(s)ds = f(t)$, $t>0$ $\iff$ $u(t) = f'(t)$ soln.

Fredholm has stuff on both sides of diag.

Eg $\int_0^1 t^2 s\, u(s)\,ds = \frac{t^2}{3} \quad 0<t<1$.

bring out: $t^2\int_0^1 s\,u(s)ds = \frac{t^2}{3}$ so any func u sat. $\int_0^1 s\,u(s)ds = 1/3$ is a soln.

eg $u(t) = \underset{\text{particular soln}}{t} + \underset{\text{homog. soln.}}{(\text{any func} \perp t)}$

soln. highly nonunique, typ. of 1st kind.

K is rank-1 since for any u, $(Ku)(t)$ = a multiple of t^2.

Bounded operators: $\|K\| = \sup_{\|u\|=1}\|Ku\|$ for your choice of norm, eg sup (L^∞), L^2, etc.

Eg. space $= C[a,b]$ w/ ∞-norm:

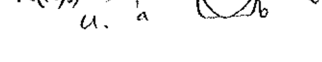
for each $t\in[a,b]$, $|(Ku)(t)| = \left|\int_a^b k(t,s)u(s)\,ds\right| \le \int_a^b |k(t,s)|ds$ if $\|u\|_\infty = 1$ — note: $|(v,u)| \le \|v\|_1\|u\|_\infty$ again

so $\|K\|_\infty \le \sup_{t\in[a,b]}|(Ku)(t)| = \sup_{t\in[a,b]}\int_a^b|k(t,s)|ds$ "biggest row-integral of abs. val of kernel"

eg $k\in C^2[a,b]$ has $\|K\|_\infty < \infty$.

Can say more: above $\le$ is $=$! Why? {Pick t_0 = the t which maximizes $\int_a^b|k(t,s)|ds$ (exists). Continuous funcs u can approximate sign $k(t_0,s)$ arb. well.

"This is NA Thm 12.5. (explicitly gives example of this).



so $\|K\|_\infty \ge \int_a^b |k(t_0,s)|ds - \varepsilon \quad \forall\varepsilon>0$.

② 1/31/12

Kernel may blow up on diagonal, eg $k(t,s) = \frac{1}{|t-s|^\gamma}$

But if $|k(t,s)| \le \frac{C}{|t-s|^\gamma}$ $\forall s,t$, $0<\gamma<1$ then L_1 norm of each row bounded,
$\Rightarrow \|K\|_\infty < \infty$ norm bnded

called 'weakly singular'. $\gamma \ge 1$ strongly singular, may be unbounded operator.
↳ rears ugly head: PDE apps.

Numerical solution method: Nyström (1930), 2^{nd} kind.

$$u(t) - \underbrace{\int_a^b k(t,s)u(s)\,ds}_{\text{quadr } Q_n \text{ w/ nodes } s_1,\dots s_n \text{ \& weights } w_1,\dots w_n} = f(t) \qquad t\in[a,b] \qquad \text{ie } (I-K)u=f$$

approx u by u_n which obeys

$$u_n(t) - \underbrace{\sum_{j=1}^n w_j k(t,s_j)u_n(s_j)}_{=:(K_n u_n)(t)} = f(t) \qquad (*)$$

where K_n is a rank-n operator approx to K
↳ its range $= \mathrm{Span}\{k(\cdot,s_j)\}_{j=1}^n$

ie $(I-K_n)u_n = f$.

Then values at nodes $u_i^{(n)} := u_n(s_i)$ sat. the lin. sys,

$$\forall i=1,\dots n,\quad u_i^{(n)} - \sum_{j=1}^n w_j k(s_i,s_j)u_j^{(n)} = f(s_i) \qquad (LS)$$

$$\text{ie} \quad (I - A)\vec{u}^{(n)} = \vec{f}$$

$\vec{u}^{(n)}$ ← vector. $\vec{f}$ ← RHS at nodes vector.
A: $n\times n$ matrix, $A_{ij} = k(s_i,s_j)w_j$

- So you've solved for u at nodes — how get back full func $u_n(s)$? Lagrange interp. poss; better is: since exact

Thm (12.11) If any vector $\{u_i^{(n)}\}_{i=1}^n$ is soln. to (LS), then $u_n(t) = f(t) + \sum_{j=1}^n w_j k(t,s_j)u_j^{(n)}$ solves $(*)$, exactly ← surprising that we have exact interpolation. ← call formula (N). for Nyström interpolant.

Pf: → Don't try to insert (N) into $(*)$!
$u_n(s_i) = u_i^{(n)}$ $\forall i$, since set $t=s_i$ in (N), gives (LS)
Use this to sub for $u_j^{(n)}$ in (N) turns it into $(*)$, $\forall t$. Subtle!

- $(*)$ expresses u_n as f + span{ column slices of kernel at node }
↳ $k(\cdot,s_j)$, form interpolation basis.

k:

- (LS) is equiv. of Vandermonde sys. requiring interpolant agrees at nodes; (N) is interpolation formula
- if drop I, can apply to 1^{st} kind, but there is no interpolation formula (N) now, just get $\{u_j^{(n)}\}_{j=1}^n$.

Note: $\|K_n - K\| \not\to 0$ as $n\to\infty$ not convergent in norm topology (shame since would be easy to prove stuff).

But do have ptwise convergence $\|(K_n-K)\phi\| \to 0$ as $n\to\infty$ (← can still prove stuff!)
ie for each $\phi \in C([a,b])$