Math 126 Winter 2012: Rough lecture notes

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1 Introduction

Numerical mathematics is at the intersection of analysis (devising and proving theorems), computation (devising algorithms, coding efficiently), and addressing application areas (e.g. PDE problems in engineering, science, technology).

This course will focus on the first two: analysis, and coding/testing computer algorithms. What is numerical analysis? Trefethen [1] gives an inspiring answer: it is not merely the study of rounding errors in computations, rather, it is the study of algorithms for the problems of continuous mathematics. We should also remind ourselves that carelessness over rounding errors, and over convergence issues, in numerical algorithms has caused loss of life and equipment destruction with losses of $\$10^8$ (see Arnold disasters website). Our goal is to understand the mathematics behind our algorithms, and be able to code them reliably and invent new ones.

Our topic is the solution of PDEs via integral equations (IEs). Along the way we touch upon rounding error, quadrature, numerical linear algegra, convergence, etc.

Paradigm PDE: Let $\Omega \subset \mathbb{R}^2$ be an open connected domain. (All of this works in higher dimensions too.) The interior BVP for Laplace's equation is

$$\Delta u = 0 \text{ in } \Omega \quad (1)$$

$$u = f \text{ on } \partial\Omega \quad (2)$$

where $\partial\Omega$ denotes the boundary of the set Ω, i.e. the set of points that are both limit points of sequences in Ω and in $\mathbb{R}^2$
Omega. The 'boundary data' is the given function f on $\partial\Omega$. Applications include electrostatics (u represents electric potential), steady-state heat distribution (u is temperature), complex analysis (u is the real part of an analytic function), and Brownian motion or diffusion (u is probability density).

Paradigm IE: Let $[0, 1]$ be an interval, and we are given $f \in C([0, 1])$, and $k \in C([0, 1]^2)$ i.e. a continuous function on the unit square. Then find a function u satistfying the integral equation

$$u(t) + \int_0^1 k(t, s)u(s)ds = f(t) \qquad \text{for all } t \in (0, 1) \quad (3)$$

This is a Fredholm equation, and since u itself is present on the LHS, is called '2nd kind'.

To give an idea of the intimate connection between the above BVP and IE, consider that uniqueness for the BVP is easy to prove: Let u and v be solutions, then $w = u - v$ satisfies $\Delta w = 0$ in Ω, and $w = 0$ on $\partial\Omega$. But by the maximum principle, the maximum of w over Ω cannot exceed the maximum on $\partial\Omega$, which is zero. The same holds for $-w$, so $w \equiv 0$, and we have uniqueness. In contrast, *existence* of a solution to the BVP is much harder. It was first proved by transformation of the BVP to an IE, in 1900 by Fredholm, and, along with Hilbert's work that decade, became the foundation of modern functional analysis. Here the identification is made between the 1D sets $\partial\Omega$ and $[0, 1]$. Thus the IE becomes a *boundary integral equation* or BIE.

The beautiful thing is that this method of proof leads to an efficient numerical method for solving the BVP. Crudely speaking, the efficiency stems from the reduction in dimensionality from u being an unknown function in 2D in the BVP to only in 1D in the IE.

Waves: As well as Laplace, we will also study the Helmholtz equation

$$(\Delta + \omega^2)u = 0 \tag{4}$$

where $\omega > 0$ is a frequency. What do solutions of this look like? The 1D analog is the ODE $u'' + \omega^2 u = 0$ which has solutions such as $\sin \omega x$ or $e^{i\omega x}$ which oscillate with wavelength $2\pi/\omega$. Similar things happen in higher dimensions, except that waves may travel in all directions. See picture

Notice that Laplace and Helmholtz are both *elliptic* PDE since the signs of the 2nd derivatives are the same. The contrasts with the wave equation,

$$\tilde{u}_{xx} + \tilde{u}_{yy} - \tilde{u}_{tt} = 0 \tag{5}$$

for the time-dependent field $\tilde{u}(x, y, t)$, which could represent acoustic pressure, for example. The wave equation is *hyperbolic* since its has mixed signs of 2nd derivatives. The mnemonic is to convert derivatives to powers of the coordinate (this is actually called the 'symbol' of a differential operator; see pseudodifferential operators):

$$\begin{array}{lclcl} u_{xx} + u_{yy} = 0 & \leftrightarrow & x^2 + y^2 = \text{const} & \leftrightarrow & \text{ellipse (here happens to be a circle)} \\ u_{xx} - u_{yy} = 0 & \leftrightarrow & x^2 - y^2 = \text{const} & \leftrightarrow & \text{hyperbola} \end{array}$$

Equations such as the heat equation have no 2nd-derivative in one of the variables, and are thus parabolic. Given even rough boundary data, elliptic PDEs lead to very smooth (even sometimes analytic) solutions; on the other hand, with hyperbolic PDEs rough initial data is carried along characteristics and remains nonsmooth. The picture for the wave equation is of the light cone disturbance produced by point-like initial data at the origin at $t = 0$.

The Helmholtz equation follows from the wave equation when the assumption of motion in time at a single frequency is made, e.g. if I were to sing in this room with a pure tone at a single frequency, the pressure field would settle into one with 'harmonic' time-dependence

$$\tilde{u}(x, y, t) = u(x, y)e^{-i\omega t}$$

Substitution of this into (5) and canceling exponential factors gives (4).

When waves traveling in free space hit an obstacle this is a scattering problem. One then needs to solve an exterior problem, with (4) holding in the unbounded domain $\mathbb{R}\ \overline{\Omega}$, with given boundary data as before, and a so-called 'radiation condition'.

What BIE methods are good for: Piecewise-homogeneous media, i.e. the coefficients of the PDE are constant in chunks of space touching on lower-dimensional boundaries. BIEs are excellent especially for exterior problems, finite element methods cannot easily handle the infinite extent of the domain. Also, BIE are excellent for high frequencies $\omega \gg 1$, since then there are many wavelengths across the domain, and the lower dimensionality of BIE vs FEM is a huge advantage.

What BIE methods are not good for: Variable-coefficient PDEs, or nonlinear PDEs. Note that there are IE methods for some of these, namely, volume-integral based methods such as Lippman-Schwinger.

2 Numerical Linear Algebra: Stability and Conditioning

Well, now we go over to scanned paper lectures. . .

(One day I will TEX up the whole thing)

References

[1] L. Trefethen. The definition of numerical analysis. *SIAM News*, November 1992. `http://people.maths.ox.ac.uk/trefethen/essays.html`.