Math 126 WORKSHEET : Condition # of basic operations

1/12/12
Barnett.

Jacobean matrix.

Relative condition # $\kappa := \sup_{\delta x} \dfrac{\|\delta f\|/\|f\|}{\|\delta x\|/\|x\|} = \dfrac{\|J(x)\|_2}{\|f\|/\|x\|}$

Find κ for the 'problems':

A) $f(x) = x^{\alpha}$ [Hint: $m=n=1$ so find J which is 1×1 matrix]

B) $f(x) = 1+x$

Can it ever be ill-conditioned? When?

C) $f(x_1,x_2) = x_1 - x_2$ [Hint: $n=2$, $m=1$ so J is 1×2 matrix]

Can it be ill-cond? When?

D) $f(x) = \sin x$ [Hint: you can give a lower bnd on κ]

Say $|x| < 10^3$: is it well-cond?

Say $x = 10^{100}$: " " ?

Math 126 WORKSHEET : Condition # of basic operations — SOLUTIONS

1/12/12 Barnett.

Relative condition # $\kappa := \sup_{\delta x} \frac{\|\delta f\|/\|f\|}{\|\delta x\|/\|x\|} = \frac{\|J(x)\|_2}{\|f\|/\|x\|}$ ← Jacobian matrix.

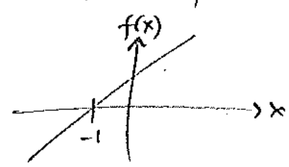
Find κ for the 'problems':

A) $f(x) = x^\alpha$ [Hint: $m=n=1$ so find J which is 1×1 'matrix']

$J = f' = \alpha x^{\alpha-1}$

so $\kappa = \frac{|J|}{|f|/|x|} = \frac{|\alpha|\,|x^{\alpha-1}|}{|x^\alpha|/|x|} = |\alpha|$ well-cond for power $|\alpha| \lesssim 10^3$.

B) $f(x) = 1+x$ $J = 1$, $\kappa = \frac{|x|}{|f|} = \left|\frac{x}{1+x}\right| \to \infty$ as $x \to -1$.



Can it ever be ill-conditioned? When? @ $x=-1$, $\kappa = \infty$ but note $\hat{\kappa} = 1$ still (abs cond #).

C) $f(x_1,x_2) = x_1 - x_2$ [Hint: $n=2$, $m=1$ so J is 1×2 matrix]

$J = [1 \ -1]$ $\|J\|_2 = \sqrt{2}$ $\kappa = \frac{\sqrt{2}}{|x_1-x_2|}\sqrt{x_1^2+x_2^2}$

yes, when $x_1 \approx x_2$. . More precisely

Can it be ill-cond? When?

D) $f(x) = \sin x$ [Hint: you can give a ~~lower~~ bound on κ] no: this was misleading

$\kappa = \frac{|\cos x|}{|\sin x|/|x|} = |x \cot x|$ $x\to 0$ $x\cot x \to 1$ so $\kappa = 1$.

Say $|x| < 10^3$: is it well-cond? yes unless $\cot x \to \infty$ ie $x \to n\pi$, $n \in \mathbb{Z}\setminus\{0\}$.

Say $x = 10^{100}$: " " ? unless $\cot x$ extremely small, $\kappa \sim |x| \sim 10^{100}$

this is because of phase uncertainty: