Solving Helmholtz problems with a basis of fundamental solutions: the role of singularities in the analytic continuation

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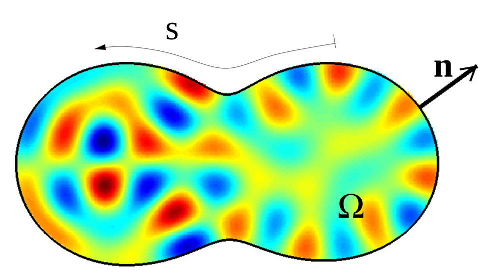
Abstract

The method of fundamental solutions (MFS) has been successfully used for solving wave problems: Greens function sources are placed outside the domain of interest and their coefficients adjusted to match desired boundary conditions. For example, in conjuction with the scaling method, one can use it to compute high-frequency eigenmodes with unprecedented efficiency [1], with applications including quantum chaos. However, there is currently little understanding of how to choose the source locations, and poor choices lead to an unusable method. We analyse this in analytic domains, proving spectral convergence in the disc, and showing numerically similar behavior in non-convex analytic domains [2]. We demonstrate that it is singularities in the analytic continuation of the solution field that control stability, in particular the coefficient sizes, in the algorithm. This enables us to develop a method for source location which adapts to the singularities induced by the so-called Schwarz function of the domain. We also show that MFS is highly competitive with boundary integral methods while possessing the advantage that the field may be accurately evaluated up to the boundary.

Helmholtz BVP



$$\Delta u + k^2 u = 0 \quad \text{in } \Omega \qquad (1a)$$
$$u = v \quad \text{on } \partial\Omega \qquad (1b)$$

Unique solution $\Leftrightarrow$ k^2 not a Dirichlet eigenvalue

High frequency (k) $\rightarrow$ seek boundary formulation

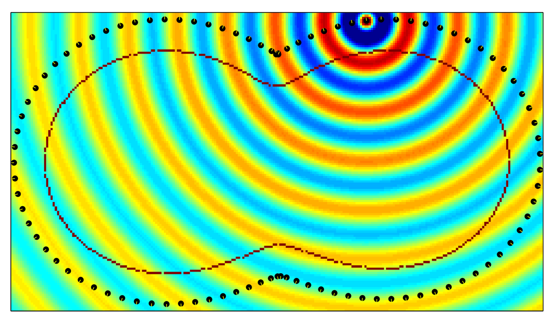
$\Omega \subset \mathbb{R}^2$ simply connected, analytic boundary $\partial\Omega$

Given approx. soln. $\tilde{u}$, boundary error norm $t[\tilde{u}] := \|\tilde{u} - v\|_{L^2(\partial\Omega)}$ controls interior error norm:

$$\|\tilde{u} - u\|_{L^2(\Omega)} \;\leq\; \frac{C_\Omega}{d}\|\tilde{u} - v\|_{L^2(\partial\Omega)} \qquad \text{(Moler-Payne 1968)}$$

where $d := \min_j |k^2 - E_j|/E_j$ is 'distance' to nearest Dirichlet eigenvalue E_j.

Method of Fundamental Solutions



Given N charge points $\mathbf{y}_j$ outside Ω, approximate

$$\tilde{u} = u^{(N)}(\mathbf{x}) = \frac{i}{4}\sum_{j=1}^{N} \alpha_j H_0^{(1)}(k|\mathbf{x} - \mathbf{y}_j|)$$

Find coeffs $\boldsymbol{\alpha} := \{\alpha_j\}_{j=1,\ldots,N}$ which minimize error t

This is a linear **least-squares problem**

Implementation: quadrature at equally-spaced boundary points $\{\mathbf{x}_m\}_{m=1\cdots M}$ gives linear system

$$A\boldsymbol{\alpha} = \boldsymbol{v} \qquad\qquad A \text{ is } M \times N \text{ matrix}$$

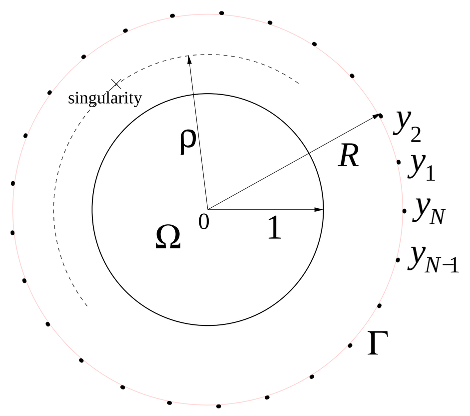
where $A_{mj} := \frac{i}{4}H_0^{(1)}(k|\mathbf{x}_m - \mathbf{y}_j|)$ and $\boldsymbol{v} := \{v(\mathbf{x}_m)\}_{j=1,\ldots,M}$. Usually overdetermined $M > N$.

For Laplace BVP ($k = 0$), MFS exponentially convergent, *i.e.* $t[u^{(N)}] \leq C\tau^{-N}$ for some $\tau > 1$:

- Eisenstat 1970's: some such sequence of $\mathbf{y}_j$ and coeffs $\boldsymbol{\alpha}$ exists
- Katsurada 1990's: holds if $\mathbf{y}_j$ chosen via conformal map of domain (exterior, annular, etc)

We now give first analytic results for Helmholtz ($k > 0$), analogous to Katsurada [4]

Analysis for the unit disc



Identify $\mathbb{R}^2$ with complex plane

Analytic boundary data $v|_{\partial\Omega}$ may be analytically continued to function $v(z)$ analytic in annulus $\{z : \rho^{-1} < |z| < \rho\}$

ρ is largest such value: distance to nearest singularity

Place charge points equally on circle Γ of radius R

Theorem 1 (exponential convergence) *Let $R > 1$, N be even, analytic boundary data v, with ρ defined above. The minimum boundary error t achievable with the MFS in the unit disc satisfies*

$$t \;\leq\; \begin{cases} C\rho^{-N/2}, & \rho < R^2 \\ C\sqrt{N}R^{-N}, & \rho = R^2 \\ CR^{-N}, & \rho > R^2 \end{cases}$$

'rough data', error due to lack of indep Fourier modes beyond frequency $N/2$ in MFS basis

'smooth data', error limited by aliasing due to discreteness of MFS charges on Γ

where each time C means a different constant which may depend on k, R, and v, but not N.

Theorem 2 (growth of coefficient norm) *Consider sequences of coefficient vectors $\boldsymbol{\alpha}$ such that the error converges as in Theorem 1 as $N \to \infty$. Then,*

- *if $R < \rho$, there exists a sequence with $|\boldsymbol{\alpha}|$ remaining bounded as $N \to \infty$*
- *if $R > \rho$, there is a constant C such that for every such sequence $|\boldsymbol{\alpha}| \;\geq\; C\sqrt{N}\left(\frac{R}{\rho}\right)^{N/2}$*

Consequence: given ρ, choose MFS radius $R \in (\sqrt{\rho}, \rho)$ for best convergence rate and no growth

Why is coefficient growth *bad*? Roundoff limits error to $t \approx \epsilon_{\text{mach}}|\boldsymbol{\alpha}|$ (double precision $\epsilon_{\text{mach}} \approx 10^{-16}$)

Proofs rely on three key steps:

1. Map from single-layer potential (charge density) on Γ to $\partial\Omega$ values is *diagonal* in Fourier basis
2. Map's eigenvalues $\hat{s}(m) = \frac{i\pi}{2}H_m^{(1)}(kR)J_m(k)$ bounded: $\frac{c_s}{|m|}R^{-|m|} \leq |\hat{s}(m)| \leq C_s R^{-|m|}$
3. Discrete charges only approximate a smooth density: aliasing (folding) errors in higher Fourier modes

High k asymptotics: can get $t \approx \epsilon_{\text{mach}}$ with only 2 degrees of freedom per wavelength (ppw) on $\partial\Omega$

General analytic domains: the Schwarz function

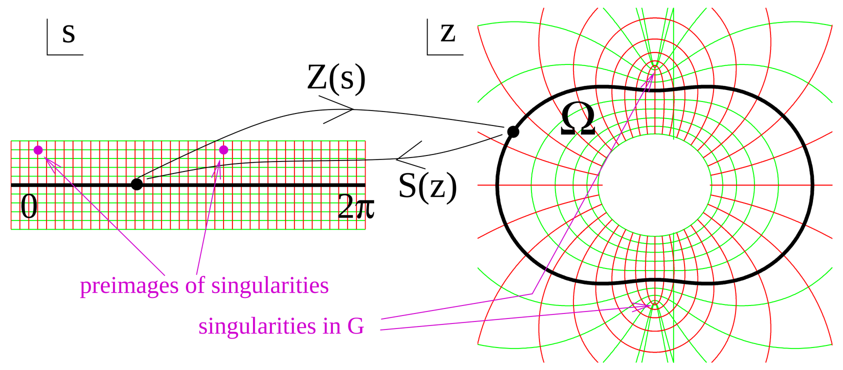
parametrize $\partial\Omega$ by map $Z(s)$, $s \in [0, 2\pi)$

Z^{-1} exists in some strip: $Z(S(z)) = z$

Ω has unique Schwarz function G:

$$G(z) := \overline{Z(\overline{S(z)})}$$

note $\overline{G(z)}$ is reflection of z in $\partial\Omega$



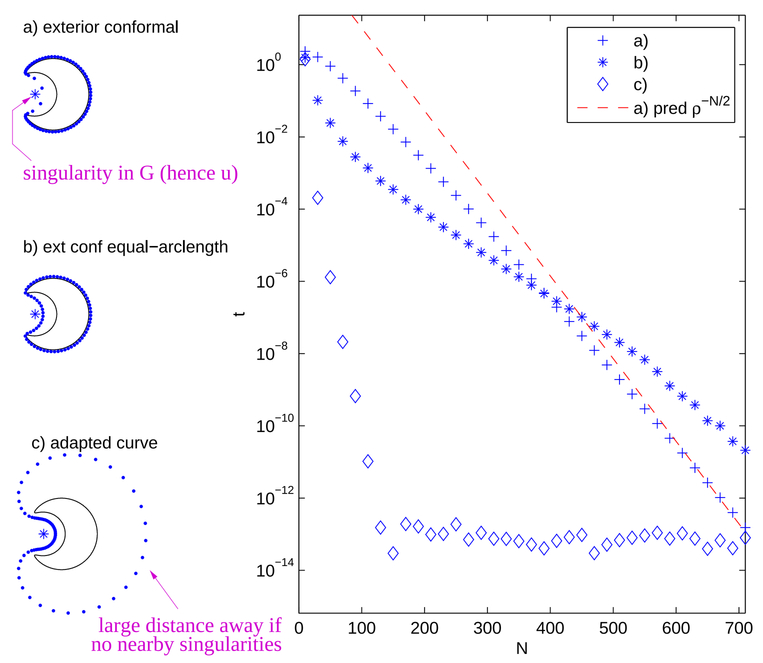
Millar [5] (via Bergman-Vekua analytic PDE theory in $\mathbb{C}^2$): singularities in analytic continuation of u generically located at singularities in continuation of v (as before) and/or G (a new twist)

Based on numerical observation [2] we have (analogous to little-known results in scattering [3]):

Conjecture 1 *Let Γ be any Jordan curve enclosing $\overline{\Omega}$, with $\text{dist}(\Gamma, \partial\Omega) > 0$, on which MFS charge points are chosen asymptotically densely. Then the coefficient norm $|\boldsymbol{\alpha}|$ that minimizes t grows asymptotically exponentially as $N \to \infty$ if and only if Γ encloses a singularity of the analytic continuation of u.*

Some numerical results

Convergence for three ways to locate the MFS charge points for a crescent domain, data $v \equiv 1$. The only singularity (due to G) lies outside Γ in each case; indeed $|\boldsymbol{\alpha}|$ does not grow as $N \to \infty$.



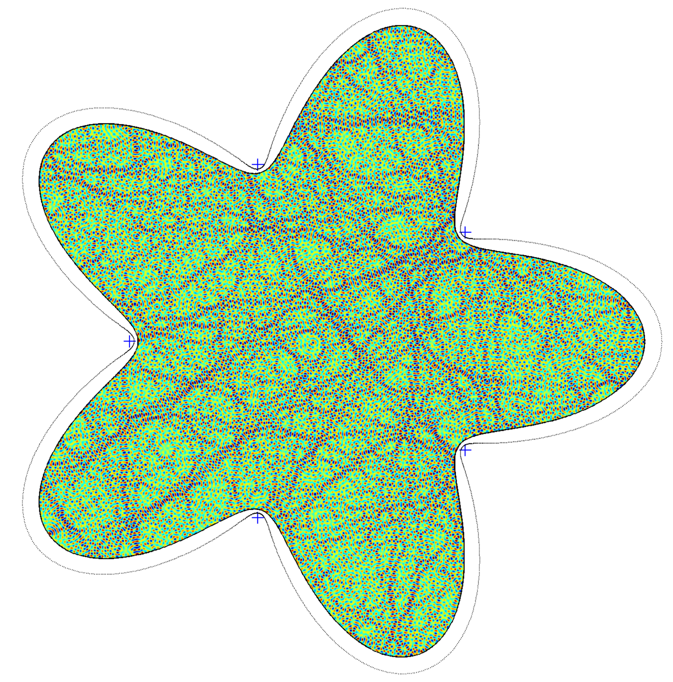
(a) and (b) use a conformal map. The convergence rate in (a) matches a conjecture [2] $t \sim \rho^{-N/2}$, where ρ is *conformal* distance to nearest singularity.

In (c) we adapt Γ to the s-plane singularities $\{t_\sigma + i\tau_\sigma\}_{\sigma=1,\ldots,n_{\text{sing}}}$ by defining $s(\chi) = t + iy(t)$, for $t \in [0, 2\pi)$, where

$$\frac{1}{y(t)} = \frac{|Z'(t)|}{D_{\text{max}}} + \sum_{\sigma=1}^{n_{\text{sing}}}\left[\gamma\tau_\sigma + \beta\frac{1 - \cos(t - t_\sigma)}{t_\sigma}\right]^{-1}$$

with parameters $\beta = 0.7, \gamma = 0.4$, and D_{max}. Charge points have spacing proportional to y. This brings Γ close to $\partial\Omega$ near singularities.

High-wavenumber efficiency: solution to (1) for polar boundary $r(\theta) = 1 + 0.3\cos(5\theta)$, at k=400, data $v(z) = \text{Re}\,(z - 1 - 0.5i)^{-1}$ Note: G has 5 nearby singularities



At only $N = 3.3$ ppw, boundary error norm $t = 4 \times 10^{-10}$, pointwise agreement to 12 digits with spectral quadrature boundary integral equation (BIE).

For same error:	MFS adapted curve	BIE method
N, M	1900, 2800	3200
CPU time ($\boldsymbol{\alpha}$)	55 sec	90 sec

Conclusions

High-accuracy solution of the Helmholtz BVP requires that coefficients $\boldsymbol{\alpha}$ remain $O(1)$, which in turn requires that the MFS charge curve enclose no singularities in the analytic continuation of the solution. We prove this, with convergence rates, in the disc. We devise a singularity-adapted charge curve for general analytic domains, and show this can exceed the efficiency of layer potential methods. High-impact open questions include,

- Can the above conjectures be proven? What is highest possible convergence rate?
- Can MFS reliably solve scattering problems in 2D and 3D with spectral accuracy?
- If Ω piecewise analytic (corners), how can we best augment with particular solution bases?

References

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