Eigenmodes and quantum chaos: Lost on the frequency axis? Check your Dirichlet-to-Neumann map!

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some work joint w/ T. Betcke, Manchester, UK
Planar Dirichlet eigenvalue problem

bounded domain $\Omega \subset \mathbb{R}^2$

$-\Delta \phi_j = E_j \phi_j$ in $\Omega$

$\phi_j = 0$ on $\partial \Omega$

‘frequency’ eigenvalues $E_1 < E_2 \leq E_3 \leq \cdots \infty$

$\{E_j\} := \sigma_D$
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- Modes of a ‘drum’: acoustics, optics, EM resonators, quantum
  paradigm for harder problems: general BCs, resonances, Maxwell...
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- Modes of a ‘drum’: acoustics, optics, EM resonators, quantum
  paradigm for harder problems: general BCs, resonances, Maxwell...
- Care about high-frequency regime: wavenumber
  \( k_j := \sqrt{E_j} \gg 1 \)
  then \( E_j, \phi_j \) expensive to compute by standard methods
Two classes of methods for eigenmodes

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- **Local basis representation**
  - *e.g.* polynomials in elements
  - basis satisfies BCs, not the PDE
  - solve: sparse matrix eigenvalue

- **Global basis representation**
  - *e.g.* layer potentials, plane waves
  - basis satisfies PDE (Helmholtz)
  - solve: dense matrix eigenvalue
Two classes of methods for eigenmodes

Direct discretization (mesh)
finite differencing
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\[ N = O(k^2) \quad k = \text{wavenumber} \]

Boundary methods (meshless)
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\( k \gg 1 \): boundary methods excel. **Bottleneck:** basis depends on freq \( E \)

⇒ must locate each \( E_j \) one-by-one as minima of some func \( t(E) \)
Outline

1. Method of Particular Solutions
   *searching on the frequency axis for each eigenvalue*

2. Scaling method
   *using Dirichlet-to-Neumann map to break the bottleneck*

3. Quantum Chaos
   *high-freq asymptotics and statistics of eigenmodes*
   (a) rate of quantum ergodicity
   (b) mixed dynamics: mushroom cavity
Method of Particular Solutions

Given trial freq parameter $E > 0$:

- choose basis function set $\{\xi_i\}_{i=1}^N$ with $-\Delta \xi_i = E\xi_i$ in $\Omega$, $\forall i$

  global Helmholtz solutions, e.g. plane waves, Fourier-Bessel functions

then $u = \sum_{i=1}^N \alpha_i \xi_i$ obeys $-\Delta u = Eu$ in $\Omega$
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- if can find coeff vector $\alpha \in \mathbb{R}^N$ giving $u|_{\partial\Omega} = 0$, but $u \neq 0$ in $\Omega$
  - ... then $u$ is a mode $\phi_j$ and $E$ is its eigenvalue $E_j$
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$\Omega = \text{peanut}, j \approx 700$

- Locate each approximate eigenvalue as one minimum of $t(E)$
Boundary tension given by linear algebra

\[ t(E) = \min_{u \neq 0, u \in \text{Span}\{\xi_i\}} \frac{\|u\|_{L^2(\partial\Omega)}}{\|u\|_{L^2(\Omega)}} = \min_{\alpha \neq 0} \sqrt{\frac{\alpha^T F \alpha}{\alpha^T G \alpha}} = \sqrt{\lambda_1} \]

\[ \text{matrix elements } F_{ij}(E) = \int_{\partial\Omega} \xi_i \xi_j \, ds, \quad G_{ij}(E) = \int_{\Omega} \xi_i \xi_j \, dx \]

lowest generalized eigenvalue of \( F \alpha = \lambda G \alpha \)

use identities to push to \( \partial\Omega \)
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- Dividing by \( \|u\|_{L^2(\Omega)} \) cures normalization problem of original MPS (Fox-Henrici-Moler ’67, Betcke-Trefethen ’05)

- In practice, increase \( N \) for accuracy: \( F, G \) numerically singular … QZ fails; must regularize \( G^{-1} \) (B ’00) … or GSVD approach (Betcke ’07)
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Relative eigenvalue error bounded: \[
\frac{|E - E_j|}{E_j} \leq C_\Omega t(E)
\] (Moler-Payne ’68)

- Thm (via perturbation): bound improved by factor \(O(E^{1/2})\) (B, preprint)
Early on, MPS faired poorly due to normalization/cond-# problems

Recent cross-pollination of ideas addresses these issues
Given $\Omega$, how choose basis functions?

- Each basis func $\xi_i(x)$ is a *global* Helmholtz soln at freq param $E$
- Want a set $\{\xi_i(x)\}$ giving small $t$ error

A natural choice is **plane waves**

$$\sin(k n_i \cdot x), \quad k^2 = E$$

equivalent to **Fourier-Bessel** (Jacobi-Anger exp)

$$J_l(kr) \exp(il\theta), \quad l \in \mathbb{Z}$$
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$\partial \Omega$ analytic: *exponential convergence*, rate controlled by…

Conformal distance to nearest singularity in analytic continuation of $\phi_j$

Vekua $\leftrightarrow$ approximation in $\mathbb{C}$ by polynomials

(Betcke ’05)

However in practice often useless: *e.g.* $t$ no lower than $10^{-2}$

(PWs analysed on unit disc: Perrey-Debain ’06)
Fundamental solutions basis (MFS)

$H_0^{(1)}(k | x - y_i |)$

charge points $\{y_i\}$ outside $\Omega$

Observe: much lower achievable $t$
(Karageorghis ’00, B ’06, B-Betcke ’07)

However best $t$ error still $\gg \epsilon_{\text{mach}} \approx 10^{-16}$...

What controls
\begin{align*}
\text{convergence rate} \quad ? \\
\text{when convergence halts}?
\end{align*}

(joint work with Timo Betcke, Manchester, UK)
Singularities in continuation of eigenmodes

analytically continue mode $\phi_j$ : where are nearest singularities?

Define analytic curve $\partial \Omega$ by image of $[0, 2\pi]$ in map $Z(s)$
Singularities in continuation of eigenmodes

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Define analytic curve $\partial \Omega$ by image of $[0, 2\pi]$ in map $Z(s)$

Singularities connected to reflection across an analytic arc:

- Domain has unique Schwarz func $G(z) := \overline{Z(S(z))}$ (Davis '74)
- $\overline{G(z)}$ is then reflection of $z$ in arc $\partial \Omega$ (inside↔outside)
Singularities in Schwarz reflection map

outside

G singular

inside

images of singularities
(G' = 0)
Singularities in Schwarz reflection map

Result: (Millar ’86) (Bergman-Vekua integral operator, complexified coords)

If \((\Delta + E)u = 0\), with data \(u \equiv v\) on \(\partial\Omega\), candidates for singularities in \(u\) are: singularities in \(G\) and/or in the analytic continuation of data \(v\)

- eigenmodes have \(v \equiv 0\) so their singularities due to \(G\) alone
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We now study BVP (MPS finds modes using BVP with data \( v \equiv 0 \))
Helmholtz BVP for $\Omega = \text{disc}$

$$(\Delta + E) u = 0 \quad \text{in } \Omega, \quad u = v \quad \text{on } \partial \Omega$$

Fundamental solns basis, with nearby $v$ singularity: (via Fourier analysis)
Helmholtz BVP for $\Omega = \text{disc}$

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\begin{align*}
\text{Thm: for } \rho < R, \text{ coefficients grow as } |\alpha| & \geq C \left( \frac{R}{\rho} \right)^{N/2} \quad \ldots \text{best if } R = \rho \\
\text{Other shapes: find } |\alpha| & \text{ grows iff charge curve encloses sing. in } G \text{ or } v
\end{align*}
High wavenumbers $k \gg 1$, analytic $\Omega$

Disc analysis: asymptotic $N \to 2$ charge points per wavelength on $\partial \Omega$

Singularity-adapted algorithm to choose charge points ($N$ typ 3 ppw)

- # quadrature points on $\partial \Omega$ can be similar (*e.g.* 3 ppw if smooth)
- Compare 2 ppw to integral eqns (typ 10 ppw, or 5 for high-order)
High wavenumber Helmholtz interior BVP

165 wavelengths across
3.5 ppw
\[ \| u - f \|_{L^2(\partial \Omega)} = 10^{-11} \]
26 sec (to get \( \alpha \))

B-Betcke, *in review*
Back to eigenmodes: The Bottleneck

Reminder: to locate Dirichlet eigenvalues $E_j$ with MPS...

- ‘tension’ $t(E) := \min_{u \in \text{Span}\{\xi_i\}} \frac{\|u\|_{L^2(\partial\Omega)}}{\|u\|_{L^2(\Omega)}}$

- must search on frequency axis $E$ for each minimum of $t(E)$
- if neighbours lie close one or more can be missed

With the MPS you are ‘lost on the frequency axis’

But you can bypass this search and vastly accelerate the method...
How to do better: the Dirichlet-to-Neumann map

Consider interior Dirichlet BVP for Helmholtz eqn:

\[(\Delta + E)u = 0 \text{ in } \Omega\]
\[u = U \text{ on } \partial\Omega\]

If \(f \in L^2(\partial\Omega)\), and \(E \notin \sigma_D\),
unique solution \(u\)
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DtN operator \(R = R(E)\) takes boundary data to normal derivative:

\[\mathbf{n} \cdot \nabla u|_{\partial \Omega} =: \partial_n u = R u|_{\partial \Omega}\]

Note \(R\) unbounded in \(L^2\), for smooth domain \(R : L^2(\partial \Omega) \to H^{-1}(\partial \Omega)\) bounded
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DtN operator \(R = R(E)\) takes boundary data to normal derivative:

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Note \(R\) unbounded in \(L^2\), for smooth domain \(R : L^2(\partial \Omega) \to H^{-1}(\partial \Omega)\) bounded

Let’s take a visual tour of spectrum of operator \(R\) …

(numeral results done using MFS basis for interior BVP problem)
Eigenvalues of DtN map $R(E)$

$Rv = \mu v$ defines spectrum of $R$, depends on freq parameter $E$:

Why? At Neumann eigenvalue $\exists \ u|_{\partial\Omega}$ not identically zero but $\partial_n u \equiv 0$

Now zoom out to look at larger eigenvalues…
Larger eigenvalues of DtN map

$R$ is semi-bounded from below (finite # of negative $\mu$’s) (Friedlander ’91)

Why? At Dirichlet eigenvalue $\exists \partial_n u$ not identically zero but $u|_{\partial \Omega} \equiv 0$ ($E = E_j$ gives nonunique BVP soln) Invert and look at $\lambda = 1/\mu$ ...

accumulation at $+\infty$ (unbounded)

accumulation not shown (ignored numerically)

poles at Dirichlet eigenvalues $E_j$
Inverse eigenvalues $\lambda$ of DtN map

$\lambda R(E)v = v$: Dirichlet eigenvalues $E_j$ found where $\lambda(E) \to 0$:

- $E = $ Dirichlet eigenvalue of $\Omega$
- Interior extensions of eigenfunctions of $R$ are Dirichlet eigenfunctions of $\Omega$
- Eigenfunctions in accumulating region are evanescent (boundary wavenumber exceeds $k$)

Accumulation at $0_+$
Inverse eigenvalues $\lambda$ of DtN map

$\lambda R(E)v = v$: Dirichlet eigenvalues $E_j$ found where $\lambda(E) \to 0$:

Linearize? But slopes vary with mode number $j$, $\lambda(E)$ strongly curved

Wouldn’t it be nice if all the slopes were known a priori?
**Weighted DtN map: inverse eigenvalues $\lambda$**

Premultiply $R$ by weight function $\frac{1}{w}$, \quad $0 < w \in L^1(\partial\Omega)$, **boundary weight**

\[ \lambda \frac{1}{w} R(E) v = v \]

Remarkably, slope when $\lambda$ hits zero is: \quad $\frac{d\lambda}{dE} = 1/2E_j$, \quad $\forall j \quad$ **i.e. predictable**

As $E - E_j$ grows, extensions of eigenfuncs of $R \approx$ **dilations of $\phi_j$**

The weight \quad $w = \frac{1}{x \cdot n}$

**is special!**
Why is weight \( w = (x \cdot n)^{-1} \) special?

Helmholtz BVP solved by Poisson kernel

\[
    u(x) = \int_{\partial \Omega} \partial n(s) G(x, s) u(s) ds
\]

\( G(x, y) \) is Green’s func for Helmholtz eqn w/ Dirichlet BCs on \( \partial \Omega \)
Why is weight $w = (x \cdot n)^{-1}$ special?

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- mode expansion $G(x, y) = \sum_{j=1}^{\infty} \frac{\phi_j(x)\phi_j(y)}{E - E_j}$

Define weighted mode normal derivatives $\psi_j(s) := \frac{1}{w(s)} \partial_n \phi_j(s)$, then

$$\frac{1}{w} R(E) = \sum_{j=1}^{\infty} \frac{\psi_j \langle \psi_j, \cdot \rangle}{E - E_j}$$

sum of rank-1 operators, each with pole in $E$

$w$-weighted inner prod $\langle U, V \rangle := \int_{\partial\Omega} wUV ds$, norm $\|U\|_w^2 := \langle U, U \rangle$

(unweighted version appeared in Nachmann-Sylvester-Uhlmann ’88)
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- Lemma: let $w = (\mathbf{x} \cdot \mathbf{n})^{-1}$, then $\|\psi_m\|_w^2 = 2E_m$ (Rellich ’40)
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More subtle reason: perturbation theory in \( \varepsilon := E - E_m \) gives

\[
\frac{dv}{d\varepsilon} \bigg|_{\varepsilon=0} = \sum_{j \neq m} \frac{Q_{mj}}{E_m - E_j} \psi_j + \text{(higher order)}
\]

with ‘mode coupling’ \( Q_{ij} := \langle \psi_i, \psi_j \rangle \)
\( Q_{ij} : \text{quasi-orthogonality on } \partial \Omega \) \hspace{1cm} (B, math-ph/0601006)

Exact orthogonality in interior \( \int_\Omega \phi_i \phi_j dx = \delta_{ij} \)

But, approx orthogonality on boundary! \( Q_{ij} := \int_{\partial \Omega} \mathbf{x} \cdot \mathbf{n} \partial_n \phi_i \partial_n \phi_j ds \)

Rellich gives \( Q_{ij} = 2\delta_{ij} E_j + q_{ij} \) \hspace{1cm} \text{with } q_{jj} = 0 \)
$Q_{ij}$: quasi-orthogonality on $\partial \Omega$  

(B, math-ph/0601006)

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with $q_{jj} = 0$

- assumption (Vergini ’94): off-diag terms grow $|q_{ij}| \sim |E_i - E_j|$
- semiclassics (B-Cohen-Heller ’00): for $\Omega$ ergodic, $|q_{ij}| \sim (E_i - E_j)^2$
\(Q_{ij}:\) quasi-orthogonality on \(\partial \Omega\)  

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- Thm: (B) for all bounded piecewise-smooth \(\Omega\), \(|q_{ij}| \leq C_\Omega (E_i - E_j)^2\)

Matrix \(\tilde{Q}_{ij}\) is very small size of \(Q_{ij}\) close to the diagonal responsible for small perturbations in DtN eigenfunctions.
Scaling method

Pick freq $E$, choose good Helmholtz basis set $\{\xi_i\}$ … as with MPS

Inverse eigenvals. $\lambda$ of $\frac{1}{w} R$ are extrema of

$$\frac{\|U\|_w^2}{\langle U, \frac{1}{w}RU \rangle}$$

Rayleigh quotient
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Rayleigh quotient

insert basis representation $u = \sum_{i=1}^{N} \alpha_i \xi_i$

quotient becomes $\frac{\alpha^T F \alpha}{\alpha^T G \alpha}$, extremized via gen. eig. prob. $F \alpha = \lambda G \alpha$

matrix elements $F_{ij}(E) = \int_{\partial \Omega} w \xi_i \xi_j ds, \quad G_{ij}(E) = \int_{\partial \Omega} \xi_i \partial_n \xi_j ds + \text{transpose}$

fix an origin, $w = \frac{1}{\mathbf{x} \cdot \mathbf{n}}$; $\Omega$ star-shaped
Scaling method

Pick freq $E$, choose good Helmholtz basis set $\{\xi_i\}$ ...as with MPS

Inverse eigenvals. $\lambda$ of $\frac{1}{w} R$ are extrema of $\frac{\|U\|_2^2}{\langle U, \frac{1}{w} R U \rangle}$ Rayleigh quotient

insert basis representation $u = \sum_{i=1}^{N} \alpha_i \xi_i$

quotient becomes $\frac{\alpha^T F \alpha}{\alpha^T G \alpha}$, extremized via gen. eig. prob. $F \alpha = \lambda G \alpha$

matrix elements $F_{ij}(E) = \int_{\partial \Omega} w \xi_i \xi_j ds$, $G_{ij}(E) = \int_{\partial \Omega} \xi_i \partial_n \xi_j ds + \text{transpose}$

Once GEP solved at a given freq $E$ ...

Each small gen. eigval. $\lambda_l$ gives a Dirichlet eigval. $E_j \approx E(1 - \lambda_l)$

Eigenvec. $\alpha^{(l)}$ gives basis rep. of eigenmode $\phi_j \approx \sum_{i=1}^{N} \alpha_i^{(l)} \xi_i(E_j)$

• all modes in $O(1)$ $k$-window found (e.g. $k_j \in [99.9, 100.1]$) via single GEP
Scaling method

Fix an origin, \( w = \frac{1}{x \cdot n} \); \( \Omega \) star-shaped

Pick freq \( E \), choose good Helmholtz basis set \( \{ \xi_i \} \) \( \ldots \) as with MPS

Inverse eigenvals. \( \lambda \) of \( \frac{1}{w} R \) are extrema of \( \frac{\|U\|_w^2}{\langle U, \frac{1}{w} RU \rangle} \) Rayleigh quotient

Insert basis representation \( u = \sum_{i=1}^{N} \alpha_i \xi_i \)

Quotient becomes \( \frac{\alpha^T F \alpha}{\alpha^T G \alpha} \), extremized via gen. eig. prob. \( F \alpha = \lambda G \alpha \)

Matrix elements \( F_{ij}(E) = \int_{\partial \Omega} w \xi_i \xi_j ds \), \( G_{ij}(E) = \int_{\partial \Omega} \xi_i \partial_n \xi_j ds + \text{transpose} \)

Once GEP solved at a given freq \( E \) \( \ldots \)

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- All modes in \( O(1) k \)-window found (e.g. \( k_j \in [99.9, 100.1] \)) via single GEP
- Observe errors in \( E_j \) grow as \( O(E - E_j)^3 \) rigorous analysis ongoing
- Variant of little-known method from quantum physics (Vergini-Saraceno ’94)
Global basis functions (as in MPS)

Plane waves

Practice: fail to capture the field singularities outside $\Omega$ (coeff sizes $|\alpha| \gg 10^{16}$)
Global basis functions (as in MPS)

Plane waves

Practice: fail to capture the field singularities outside $\Omega$ (coeff sizes $|\alpha| \gg 10^{16}$)

Fundamental solutions (MFS)

Practice: excellent apart from re-entrant corners
Global basis functions (as in MPS)

Plane waves

Practice: fail to capture the field singularities outside $\Omega$ (coeff sizes $|\alpha| \gg 10^{16}$)

Fundamental solutions (MFS)

Practice: excellent apart from re-entrant corners

Corner-adapted Fourier-Bessel $J_{\beta l}(kr) \sin(\beta l \theta)$ for singular corner $\theta = \pi/\beta$, $\beta \notin \mathbb{Z}$

Scaling currently cannot handle $> 1$ singular corner
   (dilated modes must be well approximated)

Note: basis size $N \approx 2$ per wavelength ($k \gg 1$)
High wavenumber example

convex (quarter stadium; plane-wave basis)

- \( k = 10^3 \), size 300\( \lambda \), 1 sec laptop CPU time per mode, error \( t \approx 10^{-4} \)
- speedup vs MPS is \( 10^3 \) if only bdry values needed
Applications: Quantum ergodicity

Q: If ray dynamics completely \textit{chaotic} (hyperbolic) in $\Omega$... how fast do the modes $\{\phi_j\}$ become spatially uniform?
Applications: Quantum ergodicity

Q: If ray dynamics completely chaotic (hyperbolic) in $\Omega$... how fast do the modes $\{\phi_j\}$ become spatially uniform?

- compute $3 \times 10^4$ modes up to $j \sim 10^6$: a few CPU-days
- fundamental solns basis, asymptotically 3 ppw, use 4-fold symmetry

A: deviations from uniformity die asymptotically as $O(E_j^{-1/4})$

... also no exceptional modes ‘scarred’ (condensing) on periodic orbits

(B, Comm. Pure App. Math. '06)
mode numbers
\[ j = 1, 10, 10^2, 10^3, 10^4, 10^5 \]

background: random plane waves
Ergodic eigenvalues and random matrices

nearest-neighbor spacings $s_j := E_{j+1} - E_j$

Conjectured (unproven): has same pdf as in random matrices (RMT)

well-known, not well tested before in ergodic cavity

(About the Cover, Notices AMS, Jan ’08)
Application: Mushroom

Unusually simple divided phase space

(Bunimovich ’01)

ergodic rays

regular rays
Application: Mushroom  
(B-Betcke, CHAOS, Dec ’07)

Unusually simple divided phase space  
(Bunimovich ’01)

First calculation of high-freq modes: $j = 1, \ldots, 16061$

- Conjecture verified (Percival ’73):
  modes localize to either regular or chaotic region
  44% are regular (cf regular phase space frac 45%)
Dynamical tunneling in the mushroom

All ‘regular’ modes have (exponentially) small ergodic component...

- categorize modes via mass in foot: \( f_j := \int_F |\nabla \phi_j|^2 ds \)

![Graph showing count and fraction of modes vs. \( \log_{10} f_j \) and \( \log_{10} E_j \).]
Dynamical tunneling in the mushroom

All ‘regular’ modes have (exponentially) small ergodic component...

- categorize modes via mass in foot: $f_j := \int_F |\partial_n \phi_j|^2 ds$

Find numerically: occurrence of $f_j$ in given interval dies $\sim E_j^{-1/3}$
- predicted by Bessel asymptotics and heuristic leakage model
- (Bäcker et al. ’08) improved fictitious system model (w/ our numerics)
High-lying ergodic mode & boundary coords

\[ k = 499.856... \]
\[ j \sim 45000 \]
\[ N = 1300 \]

20 sec/mode (bdry)
High-lying ergodic mode & boundary coords

\[ k = 499.856 \ldots \]

\[ j \sim 45000 \]

\[ N = 1300 \]

20 sec/mode (bdry)

Husimi func on boundary: scarred
Bouncing ball modes

(ongoing, w/ A. Hassell, ANU)

seq. condensing on neutral family (region $R$): observed, but unproven

$\exists$ trivial $O(1)$ quasimode in $R$

Zworski’s challenge:
construct $o(1)$ quasimode!
(mass in wing $W$ must $> 0$)
Bouncing ball modes

seq. condensing on neutral family (region $R$): observed, but unproven

$\exists$ trivial $O(1)$ quasimode in $R$

Zworski’s challenge: construct $o(1)$ quasimode!
(mass in wing $W$ must $> 0$)

Proven: mass in $W$ has lower bnd $\mathcal{C}E_j^{-2}$ (Burq-Hassell-Wunsch ’07)

We conjecture (numerical evidence): lower bnd is actually $\mathcal{C}E_j^{-3/4}$

- curious since matches Born-Oppenheimer adiabatic approx.
- architectural clues in modes w/ low mass in wing tip $T$: radial tendency
Conclusion

Global approx by fundamental solutions (MFS)...

- Spectral convergence, understand coeff. blow-up, 2 ppw

‘Scaling’ accel by $O(k^2)$: linearizing DtN map...

- Complexity $O(k^2) = O(N^2)$ per mode (w/ dense lin. alg.)
- Fastest eigenmode method known by factor $10^3$ at high freq

Ongoing:

- MFS/MPS for scattering (w/ T. Betcke)
- photonic crystal band structure (w/ L. Greengard)
- MFS for graded-index materials: fund. soln. to $(\Delta + E + x_1)$ known

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        T. Betcke (Manchester)

Funding: NSF (DMS-0507614)

Preprints, talks, movies:
http://math.dartmouth.edu/~ahl

made with: Linux, \LaTeX, Prosper
MPS is $E$-derivative of our DtN map!

Define operator $A(E)$ by $\langle U, AU \rangle = \|u\|_{L^2(\Omega)}^2$ bounded for $E \notin \sigma_D$

MPS can be written $t(E) = \text{lowest inverse eigenvalue: } \lambda_A A(E)u = u$

Proposition: $A = -\frac{1}{w} \frac{dR(E)}{dE}$
MPS is $E$-derivative of our DtN map!

Define operator $A(E)$ by $\langle U, AU \rangle = \| u \|_{L^2(\Omega)}^2$ bounded for $E \notin \sigma_D$

MPS can be written $t(E) = \text{lowest inverse eigenvalue: } \lambda_A A(E) u = u$

**Proposition:** $A = -\frac{1}{w} \frac{dR(E)}{dE}$

Fixing $U$, write $u' := du/dE$, $u$ is interior soln. *(trick from Friedlander ’91)*

\[
(\Delta + E)u' = -u \quad \text{mult by } u \text{ and integrate}
\]

\[
\int_\Omega u(\Delta + E)u' = -\int_\Omega u^2 := -\langle U, AU \rangle
\]

LHS by Green’s identity

\[
= \int_{\partial \Omega} u \partial_n u' - \int_{\partial \Omega} u' \partial_n u - \int_\Omega u'(\Delta + E)u
\]

\[
= \int_{\partial \Omega} U \frac{d}{dE} \partial_n u = \langle U, \frac{1}{w} R'(E) U \rangle
\]

- So $A$ has 2nd-order pole at each $E_j$:
MPS is $E$-derivative of our DtN map!

Define operator $A(E)$ by $\langle U, AU \rangle = \|u\|^2_{L^2(\Omega)}$ bounded for $E \notin \sigma_D$

MPS can be written \( t(E) = \text{lowest inverse eigenvalue: } \lambda_A A(E) u = v \)

**Proposition:** \( A = -\frac{1}{w} \frac{dR(E)}{dE} \)

Fixing $U$, write $u' := \frac{du}{dE}$, $u$ is interior soln. (trick from Friedlander ’91)

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= \int_{\partial \Omega} u \partial_n u' - \int_{\partial \Omega} u' \partial_n u - \int_{\Omega} u'(\Delta + E)u
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= \int_{\partial \Omega} U \frac{d}{dE} \partial_n u = \langle U, \frac{1}{w} R'(E)U \rangle
\]

- So $A$ has 2nd-order pole at each $E_j$:

**Thm (B ’04):** \( \lambda_A(E) = \frac{1}{\| \psi_j \|^2 w}(E - E_j)^2 + O((E - E_j)^4) \)

pert. theory of $A(E)$ is easier than $R(E)$ since bnded in $L^2(\Omega)$
App: time-reversed waves in chaotic cavity

(idea of G. Bal)

Wave equation \( u_{tt} = \Delta u \) in \( \Omega \), Dirichlet BCs

- Why? Disordered media, ultrasound, unscrambling, imaging...
- Chaotic \( \Omega \) gives best resolution  
  (1-channel expt, Draeger-Fink ’99)

Eigenmodes cheap: use as basis expansion (a Victorian dream!)
- instant evolution for very long times, no dispersion errors
Numerical demonstration

basis of 4300 modes $\partial_n \phi_j$ via scaling

- intermediate times (and unused locations) need not be computed!

Issues: improve upon Draeger-Fink random-wave theory...