Slender and close: accurate Stokes flows for rigid particles in challenging geometries

Alex Barnett[1]
work with Anna Broms[2], Anna-Karin Tornberg[2] and Dhairya Malhotra[1]

MIT Distinguished Seminar in CS&E, 4/17/25

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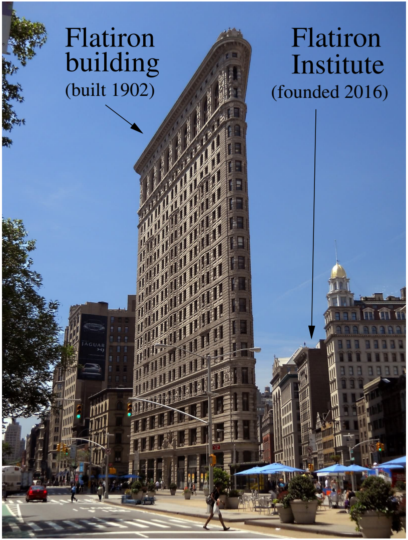
Flatiron Institute and the Center for Computational Math



A division of the Simons Foundation

CCM is one of five centers

others: biology
astrophysics/astronomy
quantum physics
neuroscience

Each ≈ 50 researchers

CCM:
scientific computing
numerical analysis
biophysics / imaging
computational statistics
machine learning

Supports both basic research
& software development / support

My current/recent areas

- Numerical PDE   linear, but complicated geometries
  wave scattering, acoustics, heat equation
  viscous fluid flow   ← today
- Nonuniform FFTs and other signal processing
  lead developer of well-used software package: FINUFFT
  CPU+GPU. Inverse Fourier imaging (MRI, black hole), GP regression
- Various scientific computing / numerical analysis
  electronic structure (quantum)
  oscillatory differential equations
  bio-math (bio-mechanics, perfusion)
  protein imaging (cryo-EM), sampling (HMC)

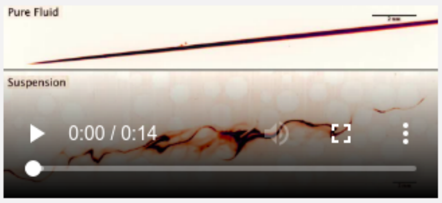
Flatiron is a problem-rich environment...
...CCM is a hub for numerical advice / tools / collaboration

Stokes flows with rigid bodies: motivations

Rheology: how do suspended particles change the effective viscosity?

Reynolds # Re=0: fluid inertia negligible

Transport in shearing suspension:



(Souzy et al, '17)

mixing, super-diffusion, turbulence, at Re ≈ 0

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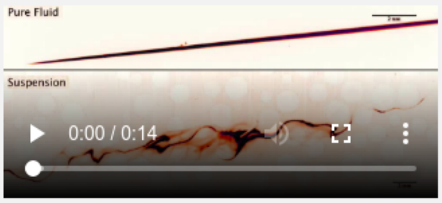

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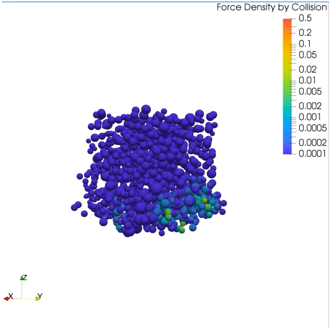

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$\sim$ 2000 cores, 2 min per time-step

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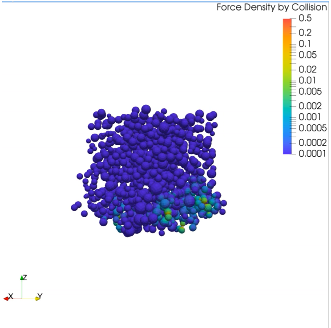

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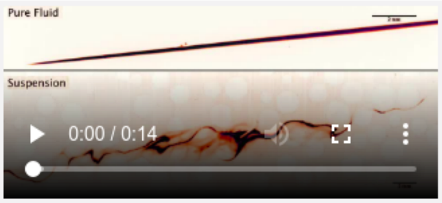

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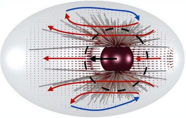

(Nazockdast et al '15)

cell biomechanics simulations (actin, nuclei)

active fluids: cilia, swimmers nonzero slip velocity $\boldsymbol{v}_{\text{slip}}$

Outline of today

1. Incompressible Stokes problems

   resistance and mobility; challenges with existing methods

2. Method of fundamental solutions (MFS)

   solve resistance and mobility for large collections of simple objects

3. Close-to-touching spheres

   augmenting MFS using pair-wise lines of images

4. Slender bodies

   replace common asymptotic approx. by *convergent* boundary integral equations (BIE)

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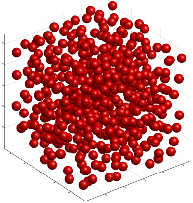
Goals throughout: tool-building. Design solvers that are...

- convergent — solves the actual model with arbitrarily small error
- high order accurate — effort grows weakly as error vanishes
- close to linear scaling in problem size — number of particles or surface unknowns

Stokes rigid-body boundary value problems

assume: 1) Reynolds # Re ≈ 0 e.g. μm particles in H_2O

2) no Brownian (thermal) forcing i.e. deterministic



Stokes Dirichlet BVP: solve for $(\boldsymbol{u}, p)$ vel, pressure

in $\Omega :=$ exterior of bodies $\Omega_1, \dots, \Omega_B$

$$
\begin{aligned}
-\mu \Delta \boldsymbol{u} + \nabla p &= \mathbf{0} \quad \text{in } \Omega \quad && \text{local fluid force balance} \\
\nabla \cdot \boldsymbol{u} &= 0 \quad \text{in } \Omega && \text{incompressible} \\
\|\boldsymbol{u}(\boldsymbol{x})\| &\to 0, \quad \|\boldsymbol{x}\| \to \infty && \text{decay condition} \\
\boldsymbol{u} &= \boldsymbol{u}_{\text{BC}} := \boldsymbol{v}_b + \boldsymbol{\omega}_b \times (\boldsymbol{x} - \boldsymbol{c}_b) \quad \text{on } \partial\Omega_b && \text{given rigid-body motion}
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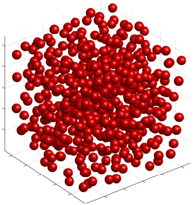

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Then from solution can extract net forces and torques on bth body:

$$\boldsymbol{F}_b = \int_{\partial\Omega_b} \boldsymbol{T} dS_{\boldsymbol{y}} \qquad \boldsymbol{\tau}_b = \int_{\partial\Omega_b} (\boldsymbol{y} - \boldsymbol{c}_b) \times \boldsymbol{T} dS_{\boldsymbol{y}}$$

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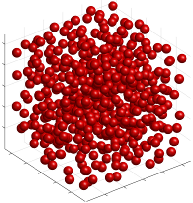

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$$\nabla\cdot\boldsymbol{u} = 0 \quad \text{in } \Omega$$ incompressible

$$\|\boldsymbol{u}(\boldsymbol{x})\| \to 0, \quad \|\boldsymbol{x}\| \to \infty$$ decay condition

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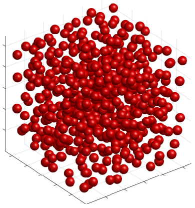

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"Mobility" problem: its inverse formally $\boldsymbol{U} = \mathcal{R}^{-1}\boldsymbol{F}$, but don't compute that way!

Challenges

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    moving geometry

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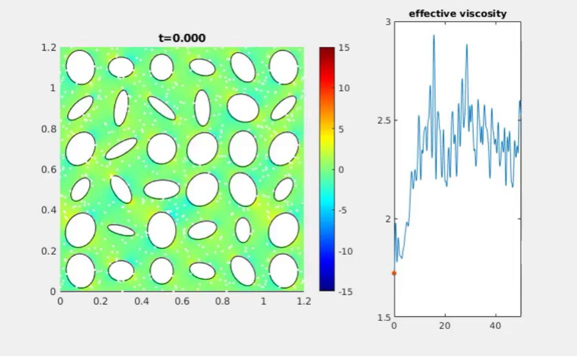
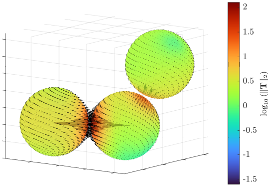
- Close-touching (dense suspensions): strong **lubrication** forces





surface separation $\delta \ll 1$

3D traction (force) peak $\sim \delta^{-2}$

spatial width $\sim \delta^{1/2}$

2D rheology of 25 neutrally-buoyant rigid ellipses, Wang–Nazockdast–B JCP '21

pressure near-singularities, diagonal force chains forming and breaking up

Various flavors of solver

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- "Stokesian Dynamics" (chem. eng. community)
  - approximate long-range hydro. via "RPY tensor" (Brady–Bossis '88 etc)
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Collisons? prevented explicitly, with an artificially inflated radius e.g. $1.1R$:(

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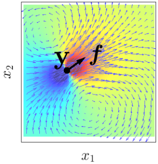

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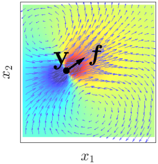

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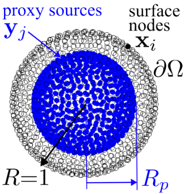

Choose "proxy" sources discretizing an interior surface

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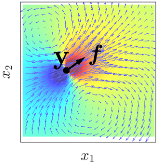

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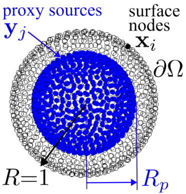

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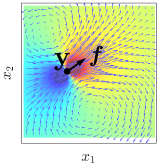

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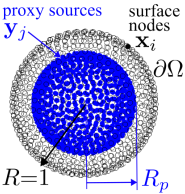

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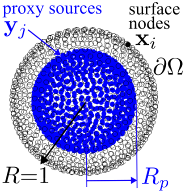
If proxy surface *well-separated* from $\partial\Omega$:

1) functions $\mathbb{S}(\cdot,\boldsymbol{y}_j)$ on $\partial\Omega$ are **smooth**: simple quadrature e.g. $M \approx 1.3N$

2) representation accurate for flow eval. **arbitrarily close** to and on $\partial\Omega$

contrast BIE: technical singular quadratures needed on $\partial\Omega$, near-singular ones close to $\partial\Omega$

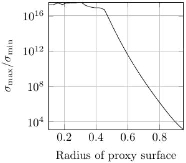
...but MFS does have a catch!



Condition # $\kappa(G) := \frac{\sigma_{\max}}{\sigma_{\min}} \gtrsim 10^8$

grows (root) exponentially in N
growth rate worse as proxy R_p shrinks
but small R_p needed for small errors!

Fixed $N = 700$:



Intuitively: $\sigma_{\min}$ controlled by the degree-$\sqrt{N}$ sph. harmonic which decayed like $(R_p/R)^{\sqrt{N}}$

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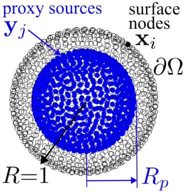

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grows (root) exponentially in N

growth rate worse as proxy R_p shrinks

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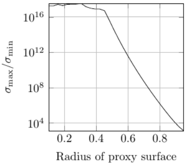

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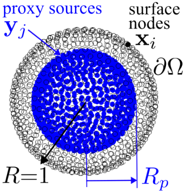

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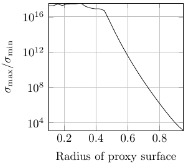

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Direct soln via SVD: $G = U\Sigma V^T$, where $\Sigma = \text{diag}\{\sigma_k\}$ — cost $\mathcal{O}(N^3)$

pick cut-off ϵ (e.g. 10^{-12}), set $\Sigma^+ := \text{diag}\{\sigma_k^{-1}$ if $\sigma_k > \epsilon\sigma_1$, else $0\}$

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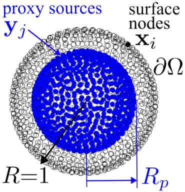

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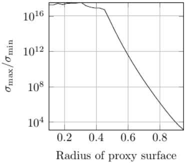

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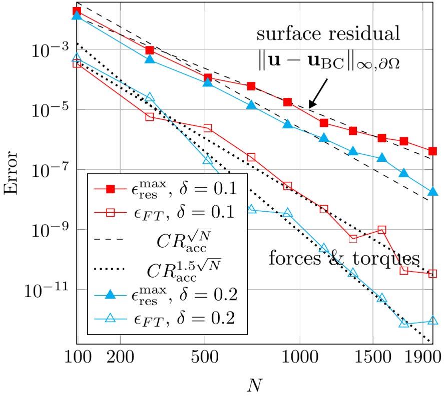
then $\boldsymbol{\lambda} \approx V\Sigma^+(U^T\boldsymbol{u}_{\text{BC}}) =: G^+\boldsymbol{u}_{\text{BC}}$ — a (possibly regularized) pseudoinverse

Note: G^+ should never be filled! Backward stable iff apply the two matvecs in above order

However, MFS effective for simple smooth bodies — $N, M \sim 10^3$, SVD $\sim$ few secs

MFS small example

MFS discretization solving resistance for $B = 2$ unit spheres: $R_p = 0.6$



Separations $\delta = 0.1$, $\delta = 0.2$

Error in forces and torques hits 10^{-10} by $N \approx 1700$

convergence root-exponential for some rate $R_{\mathrm{acc}}(\delta)$

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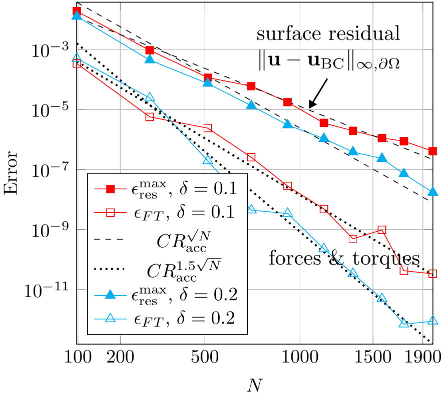

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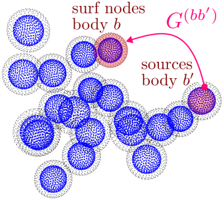
- For spheres, MFS needs same # unknowns as BIE for same error

  BIE uses elaborate vector spherical harmonic close-eval. (Corona–Veerapaneni '18)

- MFS (as BIE) can handle more general smooth shapes (Liu-B '15)

How scale to many simple bodies? ($B > 10^3$)

$$\begin{bmatrix} G^{(11)} & G^{(12)} & \dots & G^{(1B)} \\ G^{(21)} & G^{(22)} & \dots & \dots \\ \vdots & \vdots & \ddots & \vdots \\ G^{(B1)} & \dots & \dots & G^{(BB)} \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda}^{(1)} \\ \vdots \\ \boldsymbol{\lambda}^{(B)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}_{\text{BC}}^{(1)} \\ \vdots \\ \boldsymbol{u}_{\text{BC}}^{(B)} \end{bmatrix}$$



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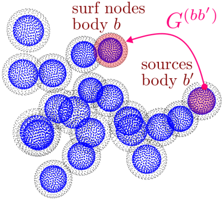

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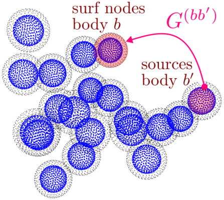

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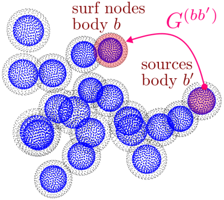

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abbrev: $\tilde{G}\boldsymbol{\mu} = \mathbf{u}_{\text{BC}}$ huge, square, well-cond. $\Rightarrow$ iter. solve (Liu–B '15, Stein–B, '22)

Iterative solution of $\tilde{G}\boldsymbol{\mu} = \boldsymbol{u}_{\text{BC}}$ for resistance prob.

We use GMRES as solver

minimizes residual in the Krylov subspace

each iteration needs matvec $\boldsymbol{u} \leftarrow \tilde{G}\boldsymbol{\mu}$

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Black-box software: `FMM3D` (Rachh et al), `PVFMM` (Malhotra), `STKFMM` (Wen et al)

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Once GMRES solved for $\boldsymbol{\mu}$, get proxy strength vector $\boldsymbol{\lambda}$ by Step 1

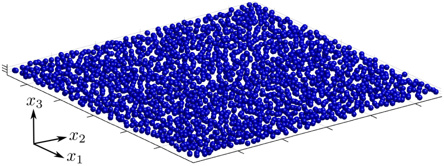
Then use MFS rep. to evaluate Stokes flow $(\boldsymbol{u}, p)$ anywhere in Ω

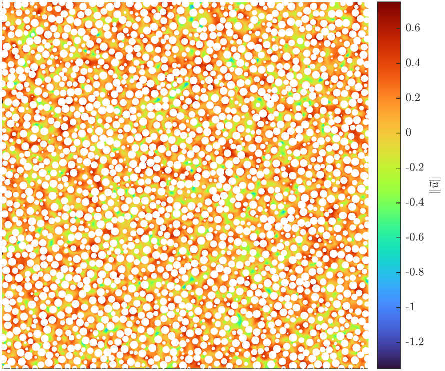
Evaluate net forces & torques: easy, sum the sources $\boldsymbol{F}_b = \sum_{j=1}^{N}\boldsymbol{\lambda}_j^{(b)}$

$\boldsymbol{\tau}_b = \sum_{j=1}^{N}(\boldsymbol{y}_j^{(b)} - \boldsymbol{c}_b) \times \boldsymbol{\lambda}_j^{(b)}$

Results: resistance problem via MFS

$B = 2000$ monodisperse sphere layer given random motions $\{\boldsymbol{v}_b, \boldsymbol{\omega}_b\}_{b=1}^{B}$:





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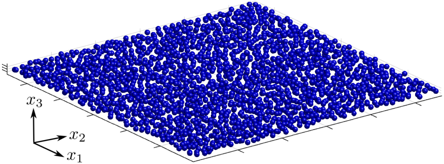

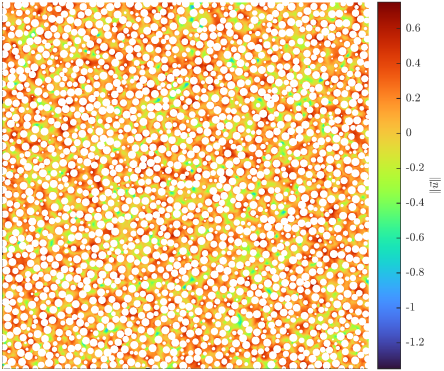

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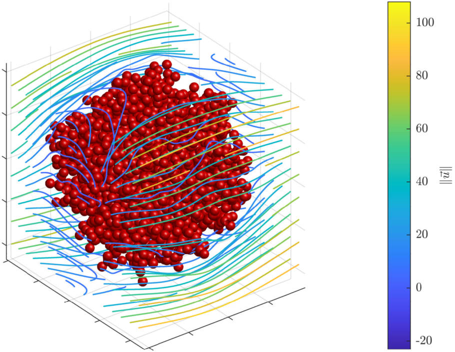
- FMM (step 2) is $> 90\%$ of cost   good
- GMRES (rel. resid. 10^{-7}): 100 iters.   weak growth w/ B
- 10^{-3} uniform residual in boundary condition $\boldsymbol{u}|_{\partial\Omega} - \boldsymbol{u}_{\text{BC}}$
- 10^{-5} in forces+torques   by convergence study

Results: cluster porosity problem via MFS

$B = 2000$ **fixed** spheres in background shear flow $\boldsymbol{u}_0 = (0, -5x_1, 0)$:



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max BC resid. error 10^{-4}

Why accurate when δ so small?!

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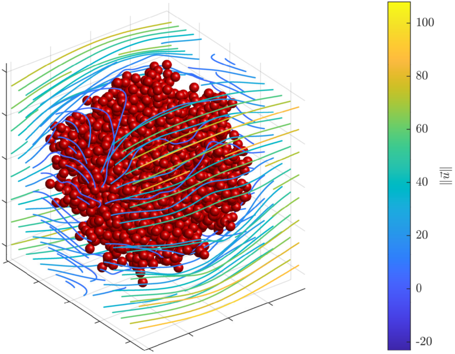

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No rel. motion $\rightarrow$ no lubrication sing.

PDE analysis needed here!

smooth $\boldsymbol{u}$, plain MFS works well

MFS for mobility problems?

Recall MFS discretization of resistance prob: (no backgnd, no slip, no precond.)

$$G\boldsymbol{\lambda} \approx \boldsymbol{u}_{\text{BC}} =: K_M U \qquad \text{motions vec. } \boldsymbol{U} := \{\boldsymbol{v}_b, \boldsymbol{\omega}_b\}_{b=1}^B \quad K_M \text{ tall: } 3MB \times 6B$$

then extract $\boldsymbol{F} := \{\boldsymbol{F}_b, \boldsymbol{\tau}_b\}_{b=1}^B = K_N^T \boldsymbol{\lambda}$ $\qquad K_N$ analogous $3NB \times 6B$

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$$\text{Mobility: } \begin{cases} G\boldsymbol{\lambda} \approx K_M \boldsymbol{U} \\ \text{s.t. } K_N^T \boldsymbol{\lambda} = \boldsymbol{F} \end{cases}$$

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How solve mobility with linear cost, as resistance? abbrev. MFS rep. $\boldsymbol{u} = \mathbb{S}\boldsymbol{\lambda}$

Let $L :=$ orthog. proj. to Col K_N. Pick $\boldsymbol{\lambda}_0$ s.t. $K_N^T \boldsymbol{\lambda}_0 = \boldsymbol{F}$

Idea: new rep. $\boldsymbol{u} = \mathbb{S}[(I - L)\boldsymbol{\lambda} + \boldsymbol{\lambda}_0]$ has correct net $\boldsymbol{F}$ since $K_N^T (I - L) = 0$

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Mobility: $\begin{cases} G\boldsymbol{\lambda} \approx K_M \boldsymbol{U} & \boldsymbol{U} \text{ unknown} \\ \text{s.t. } K_N^T \boldsymbol{\lambda} = \boldsymbol{F} & \boldsymbol{F} \text{ forces and torques given} \end{cases}$ constrained LSQ prob. in $(\boldsymbol{\lambda}, \boldsymbol{U})$?

How solve mobility with linear cost, as resistance? abbrev. MFS rep. $\boldsymbol{u} = \mathbb{S}\boldsymbol{\lambda}$

Let $L :=$ orthog. proj. to Col K_N. Pick $\boldsymbol{\lambda}_0$ s.t. $K_N^T \boldsymbol{\lambda}_0 = \boldsymbol{F}$

Idea: new rep. $\boldsymbol{u} = \mathbb{S}[(I - L)\boldsymbol{\lambda} + \boldsymbol{\lambda}_0]$ has correct net $\boldsymbol{F}$ since $K_N^T(I - L) = 0$

Idea: couple $\boldsymbol{U}$ to *unused* $6B$-dim $\boldsymbol{\lambda}$ subspace: Ansatz $\boldsymbol{U} = -K_N^T \boldsymbol{\lambda}$

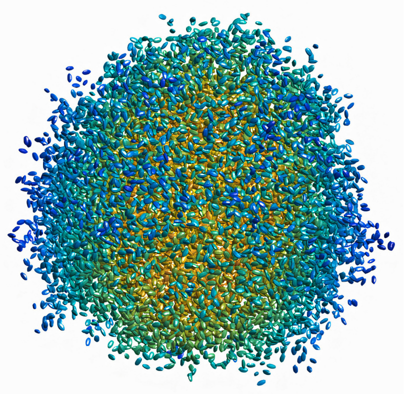
Get lin. sys: $[G(I - L) + K_M K_N^T]\boldsymbol{\lambda} = -G\boldsymbol{\lambda}_0$ known RHS, eval. by FMM

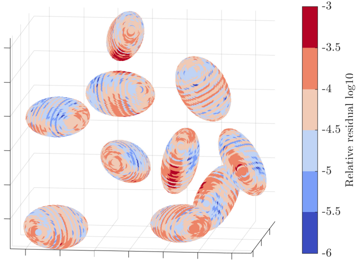
once solved, extract motions vec. $\boldsymbol{U}$ using Ansatz

- new; related to "completed double-layer" formulation (Power–Miranda '87)
- modified blocks, can apply 1-body precond, GMRES+FMM iter. solve

Large scale mobility via MFS

$B = 10000$ ellipsoids, random forces+torques: $\delta_{\min} = 0.2$, semiaxes $(0.4, 0.6, 1)$





zoom of surface resid. err. 10^{-3}

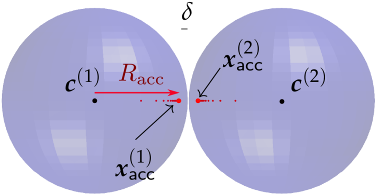
proxy sources displaced along $\boldsymbol{n}$

- 10^{-5} errors in $\boldsymbol{U}$, total degrees of freedom $3MB \approx 2.6 \times 10^7$
- solution time 2 hrs   8-core desktop, only 7 iters needed
- for mobility we see # iters. does not grow with B for resistance it grows
  - already observed (Ichiki '02, Usabiaga '16); explanation needed!
- usually mobility of this scale needs MPI (2k cores for 80k spheres: Yan et al. '20)

Close-touching spheres with relative motion

Laplace Dirichlet BVP: 2 charged conducting spheres has analytic soln

a point charge c at radius r reflects in 1 unit sphere: inversion $r \mapsto 1/r$, strength $-c/r$



Alternating sphere reflections: ∞ series

(electrostatics, Lord Kelvin 1853)

Limit (accumulation) point:

$$R_{\text{acc}} = 1 + \frac{\delta}{2} - \sqrt{\delta + \frac{\delta^2}{4}} = 1 - \mathcal{O}(\delta^{1/2})$$

limit singularity lies dist $\mathcal{O}(\delta^{1/2})$ under surface

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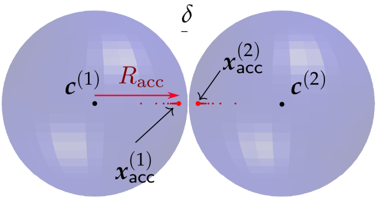

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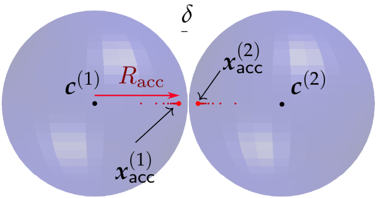

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Concept: exterior PDE soln may continue inside as PDE soln for some dist.

"analytic continuation": holomorphic theory of PDE for analytic data and boundary $\partial\Omega$

Upshot: for 2 unit Stokes spheres, $\boldsymbol{u}$ continues in to radius $R_{\text{acc}} < 1$

MFS theory: stability & convergence rate

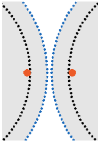
Unit spheres sep. by δ: PDE soln continues inside to $R_{\text{acc}}(\delta) < 1$

2D MFS theorems (Katsurada '89, B–Betcke '08) then *suggest* in 3D...

- stable i.e. $\|\boldsymbol{\lambda}\|_\infty = \mathcal{O}(1)$ iff $R_p \geq R_{\text{acc}}$

  proxy surface must enclose singularities

- $R_p = R_{\text{acc}}$ decent choice:



- MFS conv. rate is then set by R_{acc}:

  error $= \mathcal{O}(R_{\text{acc}}{}^{\sqrt{N}})$

This explains dashed lines in convergence plot...

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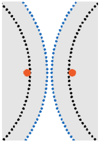

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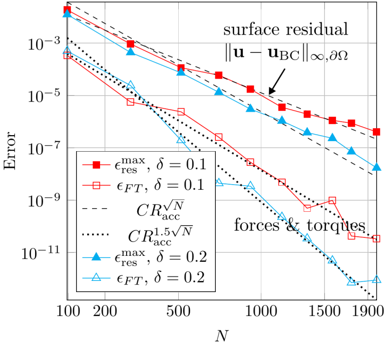

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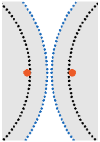

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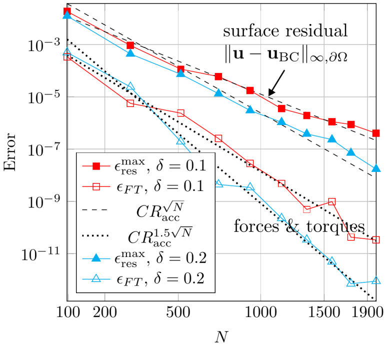


Problem: close spheres makes $R_{\text{acc}} \to 1^-$, means convergence grinds to halt

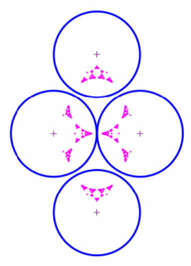
to maintain fixed error demands huge $N \gtrsim \mathcal{O}(\delta^{-1})$

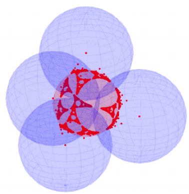
$\mathcal{O}(N^3)$ cost would be impractical for $\delta \lesssim 0.05$ We now fix that...

Augmenting MFS with images for close spheres

Multi-sphere reflection images:

2D: 3D:





2D: "Indra's pearls" (Klein 1897)

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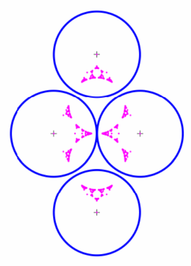

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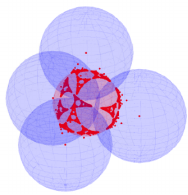

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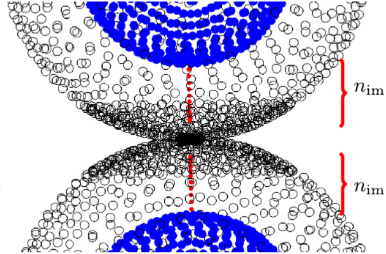
Idea: add image lines pairwise

line from proxy R_p out to R_{acc}

discretize with $n_{\text{im}} \sim 20$ nodes



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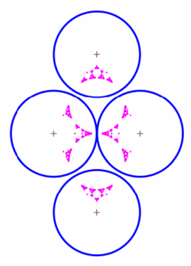

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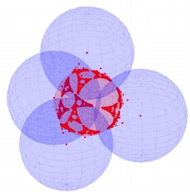

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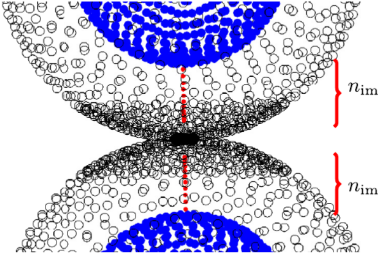

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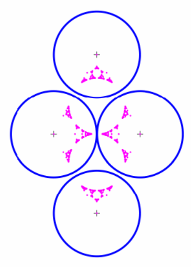

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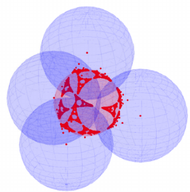

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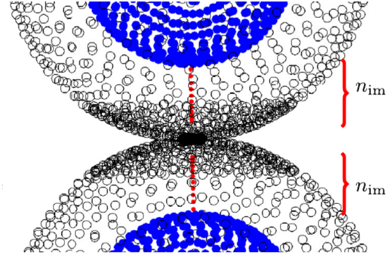

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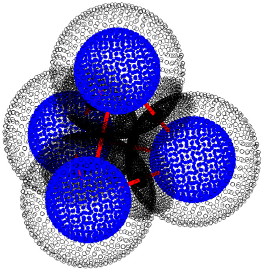
Philosophy: be *inspired* by elaborate analytic formulae

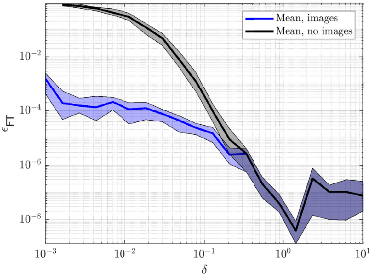
. . . but instead make the *computer* solve for the coeffs!

Results: resistance for $B = 4$ close spheres

We target 3-digit accuracy for separations down to $\delta = 10^{-3}$:

in μm particles, surface roughness breaks the model at that scale





$N \approx 700$ $n_{\text{im}} = 20$, equiv to 60 extra sources per near-contact

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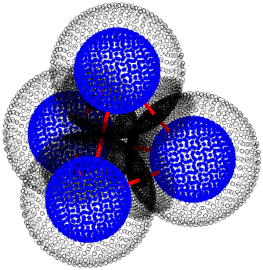
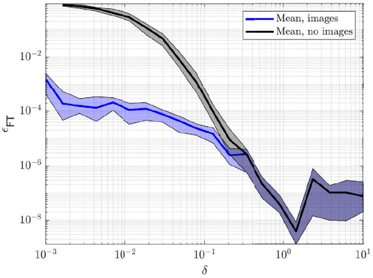




$N \approx 700$ $n_{\text{im}} = 20$, equiv to 60 extra sources per near-contact

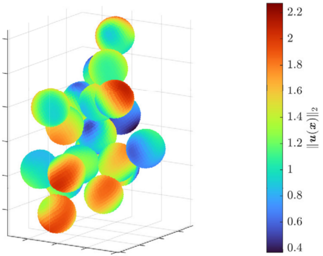
- With a few extra unknowns we've extended the δ by factor 10^2
- Compared against (non-adaptive) BIE scheme, impractical for $\delta \leq 10^{-2}$
- GMRES iter. count grows badly, like $\sim \delta^{-1/2}$ for both MFS and BIE

Results: resistance for $B = 20$ close spheres

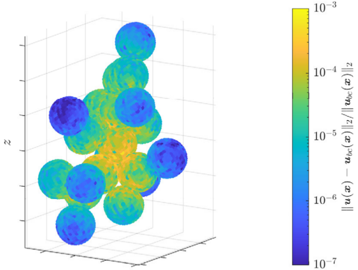
Random cluster of spheres with separations $\delta = 10^{-3}$. Target error 10^{-3}:

Random motions $\boldsymbol{U}$ excites generic lubrication singularities

surface velocity magnitudes:

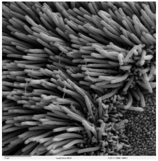


surface residuals:

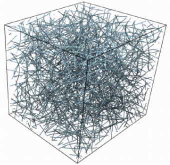


- here we used dense (slow) matvecs; drop-in of FMM in progress
- emphasize Broms ran all numerical expts and invented many ideas

Convergent BIE for slender rigid bodies (quick sketch)



trachea



(Maxian–Donev '21)

Fibers in Stokes regime:

cilia, flagellae, actin, cytoskeleton, nematic fluids, cellulose pulp

We tackle the rigid fiber case

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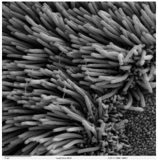

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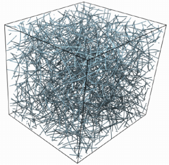

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Standard tool "slender body theory": rep. $\boldsymbol{u}$ as center-line integral

matched asymptotics in radius $\epsilon \ll 1$ (Keller–Rubinow '76, Johnson '80; Tornberg et al)

Error? classical $\mathcal{O}(\epsilon^2 \log \epsilon^{-1})$; recent rigorous analysis $\mathcal{O}(\epsilon \log^{3/2} \epsilon^{-1})$ (Mori–Ohm–Spirn '19)

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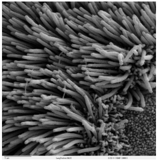

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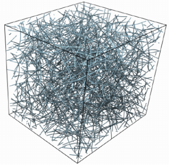

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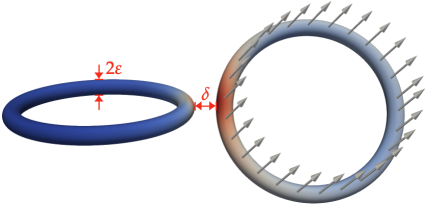
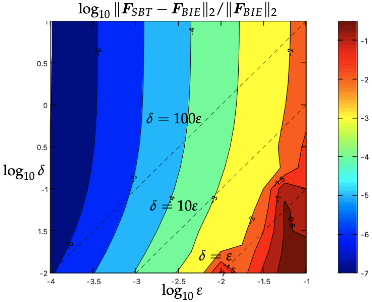
SBT inaccurate when ϵ big or $\delta <$ few ϵ:





Layer potentials and surface quadrature

Goal: high-order accurate BIE for resistance & mobility BVPs
with cost independent of ϵ user sets $\epsilon = 0.1$ or 10^{-5}, dials in accuracy, e.g. 10^{-6}

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We needed new layer potential combined representations:

$$\boldsymbol{u} = (\mathcal{D} + \frac{1}{2\epsilon \log \epsilon^{-1}}\mathcal{S})[(I - L)\boldsymbol{\sigma}] + \mathcal{S}\boldsymbol{\sigma}_0$$

mobility, stays well cond. as $\epsilon \to 0$

single-layer $\mathcal{S}[\boldsymbol{\sigma}](\boldsymbol{x}) := \int_{\partial\Omega} \mathbb{S}(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{\sigma}(\boldsymbol{y}) dS_{\boldsymbol{y}}$,

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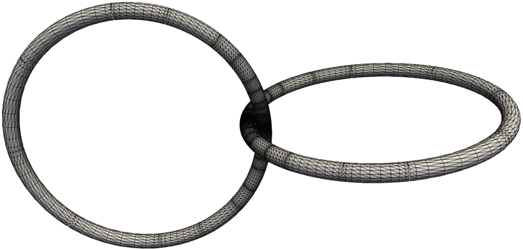
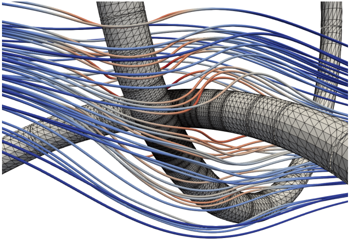
Discretize surface $\partial\Omega$: 2nd kind (I + cpt op.) $\to$ well-cond. square linear system

Chebyshev panels, adaptive in arclength, θ-Fourier modes adaptive per panel





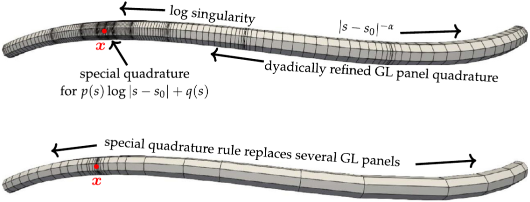
Nyström quadrature for integral operator

Integral operator on-surface is weakly singular $\mathcal{O}(1/\|\boldsymbol{x} - \boldsymbol{y}\|)$

Standard triangle/patch based BIE quadratures would die as $\epsilon \to 0$

We developed...

Outer quadrature in arclength s:



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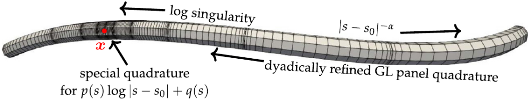
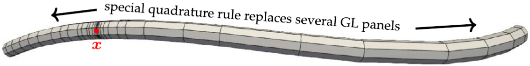
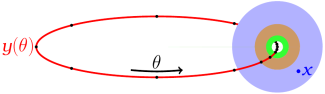




Inner quadrature in angle θ:



Both use "generalized Chebyshev quadrature": (Bremer–Gimbutas–Rokhlin '10)

precompute custom nodes & weights to integrate a family of functions

Get up to 10-digit accuracy, down to separations $\delta = \epsilon/20$ omit many details

HPC implementation SIMD kernels, OpenMP; far-field done by PVFMM (MPI)

quadrature setup rate: 20k unknowns/sec/core @ 7-digit accuracy

A large-scale result: sedimentation of 512 rigid tori

time-evolve $\dot{\boldsymbol{X}} = \boldsymbol{U}(\boldsymbol{X})$

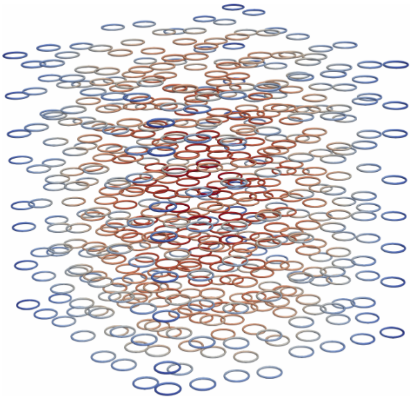
where $\boldsymbol{U}$ is mobility BVP solve at config. $\boldsymbol{X}$

gravity force, no inertia

ODE t-step: 5th-order SDC
1-body preconditioning
1.6 million unknowns
7-digit accuracy
$\sim$ 1 day on 160 cores

all Dhairya's HPC C++ codebase
plus many inventions

my job was to code SBT and prove theorems :)



Conclusions

Linear-cost solvers for Stokes flows with many simple rigid particles

- getting the right answer when close-to-touching is hard
- preconditioned fundamental solutions (MFS), pair image lines

High-aspect-ratio rigid fibers: boundary integral equations (BIE)

- new representation, high-order quadratures, HPC
- replace slender body theory in close-fiber simulations

A. Broms, A. H. Barnett, A.-K. Tornberg, *J. Comput. Phys.* **523** (2025)
A. Broms, A. H. Barnett, A.-K. Tornberg, *accepted, Adv. Comput. Math.*
D. Malhotra, A. H. Barnett, *J. Comput. Phys.* **503** (2024)
`https://github.com/dmalhotra/CSBQ`

In progress/future in this area:

- 2-body precond. for 3D mobility w/ images + FMM
- collision-handling; flexible fiber BVP
- Brownian? sampling $\boldsymbol{U}$ from Gaussian with covariance the mobility matrix

Thanks to: Mike Shelley, Leslie Greengard, Shravan Veerapaneni, Jun Wang

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EXTRAS

Preliminary results: "2-body" preconditioning

recall 1-body precond: GMRES iteration count grows as $\delta \to 0$

Idea: direct solution of a fine discretization for each near **pair** of bodies

"compress" (factorize) to an effective coarse proxy representation; GMRES sees only that

Preliminary results: "2-body" preconditioning

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Idea: direct solution of a fine discretization for each near **pair** of bodies

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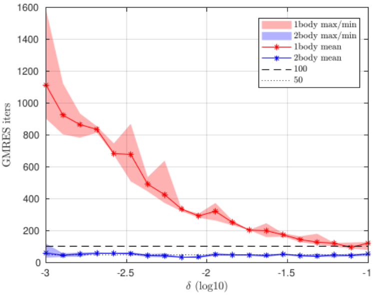
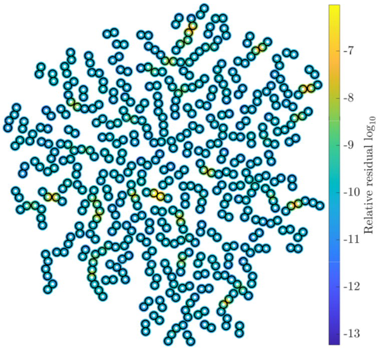
2D monodisperse disks, shows huge reduction in iteration count:

resistance:



mobility:



$\delta = 10^{-3}$. Generalizes ideas of (Cheng–Greengard '98) to local pairwise BVP solves