A Fast Convergent Boundary Integral Framework for Slender Bodies

Dhairya Malhotra, Alex Barnett

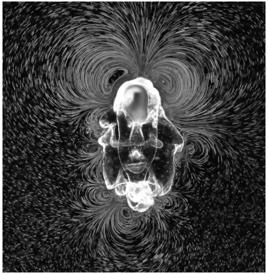
siam 2022 | Annual Meeting

July 14, 2022

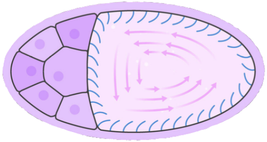
Motivations

FLATIRON INSTITUTE
Center for Computational Mathematics

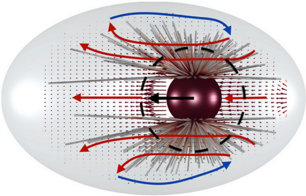
Stokes simulations with fibers are key to modeling complex fluids (suspensions, rheology, industrial, biomedical, cellular biophysics).



Starfish larvae
(Gilpin et al. 2016)



Drosophila oocyte
(Stein et al. 2021)



Mitotic spindle (Nazockdast et al. 2015)

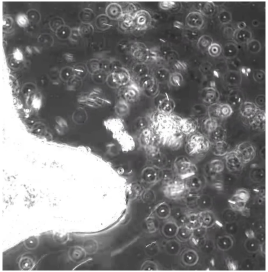
Motivations

Flatiron Institute
Center for Computational Mathematics

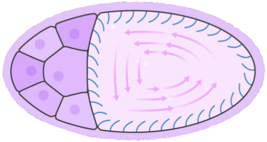
Stokes simulations with fibers are key to modeling complex fluids (suspensions, rheology, industrial, biomedical, cellular biophysics).

Slender Body Theory (SBT):

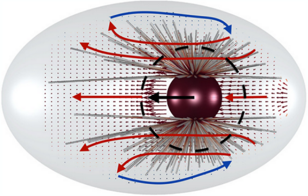
- Asymptotic expansion in radius (ε) as $\varepsilon \to 0$ (Keller-Rubinow '76).
- Doublet correction to make velocity theta-independent (Johnson '80).



Starfish larvae (Gilpin et al. 2016)



Drosophila oocyte (Stein et al. 2021)



Mitotic spindle (Nazockdast et al. 2015)

Slender Body Theory

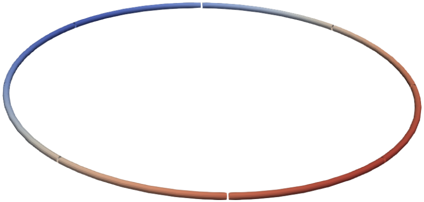
Error estimates: Rigorous analysis difficult (few very recent studies)

- classical asymptotics claims: $\varepsilon^2 \log(\varepsilon)$
- rigorous analysis: $\varepsilon \log^{3/2}(\varepsilon)$ (Mori-Ohm-Spirn '19)
- numerical tests: $\varepsilon^{1.7}$ (Mitchell et al. '21 -- verify close-touching breakdown)

  close-to-touching with gap of 10ε, only 2.5-digits in the infty-norm.

  ε=1e-2 only 1-2 digits achievable by SBT.

Sedimentation velocity of ellipse of thickness ε in a Stokesian fluid.



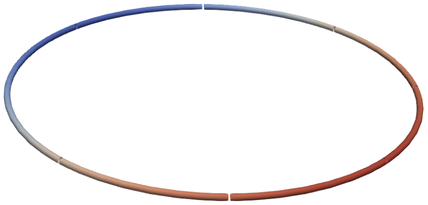
Slender Body Theory

FLATIRON INSTITUTE
Center for Computational Mathematics

Error estimates: Rigorous analysis difficult (few very recent studies)

- classical asymptotics claims: $\varepsilon^2 \log(\varepsilon)$
- rigorous analysis: $\varepsilon \log^{3/2}(\varepsilon)$ (Mori-Ohm-Spirn '19)
- numerical tests: $\varepsilon^{1.7}$ (Mitchell et al. '21 -- verify close-touching breakdown)
  close-to-touching with gap of 10ε, only 2.5-digits in the infty-norm.
  ε=1e-2 only 1-2 digits achievable by SBT.

ε	$\boldsymbol{u}_0$	Error
0.1	0.0518	$0.7e-2$
0.01	0.0736	$0.2e-3$
0.001	0.0950	$0.3e-5$
0.0001	0.1163	$0.4e-7$



Slender Body Theory

FLATIRON INSTITUTE
Center for Computational Mathematics

Error estimates: Rigorous analysis difficult (few very recent studies)

- classical asymptotics claims: $\varepsilon^2 \log(\varepsilon)$
- rigorous analysis: $\varepsilon \log^{3/2}(\varepsilon)$ (Mori-Ohm-Spirn '19)
- numerical tests: $\varepsilon^{1.7}$ (Mitchell et al. '21 -- verify close-touching breakdown)

  close-to-touching with gap of 10ε, only 2.5-digits in the infty-norm.

  ε=1e-2 only 1-2 digits achievable by SBT.

Limitations:

- no convergence analysis for fibers of given nonzero radius.
- uncontrolled errors when fibers close $O(\varepsilon)$.

Efficient convergent BIE method needed, allowing adaptivity for close interactions.

Goals

Solve the slender body BVP

- in a convergent way.
- adaptively when fibers become close.
- efficiently with effort independent of varying radius.

Validate current SBT simulations.

Most existing qudaratures cannot resolve high aspect ratio geometries.

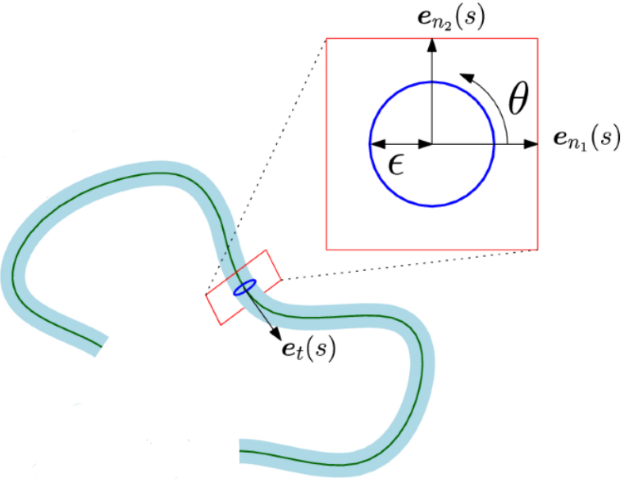
Focus on rigid fibers in this talk -- flexible fibers for future.

Related work: Mitchell et al, '21 (mixed-BVP corresponding to flexible fiber loop)

Discretization

Geometry description:

- parameterization s along fiber length
- coordinates $x(s)$ of centerline curve
- circular cross-section with radius $\varepsilon(s)$
- orientation vector $e_{n_1}(s)$



Discretization

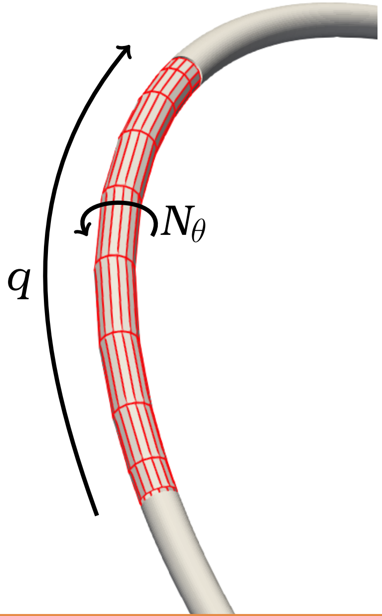
FLATIRON INSTITUTE
Center for Computational Mathematics

Geometry description:

- parameterization s along fiber length
- coordinates $x(s)$ of centerline curve
- circular cross-section with radius $\varepsilon(s)$
- orientation vector $e_{n_1}(s)$

Discretization:

- piecewise Chebyshev (order q) discretization in s for $x(s)$, $\varepsilon(s)$ and $e_{n_1}(s)$
- Collocation nodes: tensor product of Chebyshev and Fourier discretization in angle with order N_θ.



Boundary Quadratures

FLATIRON INSTITUTE
Center for Computational Mathematics

$$u(x) = \int_\Gamma \mathcal{K}(x-y)\,\sigma(y)\,da(y) = \sum_{k=1}^{N_{panel}} \int_{\gamma_k} \mathcal{K}(x-y)\,\sigma(y)\,da(y)$$

Boundary Quadratures

$$
\begin{aligned}
u(x) &= \int_\Gamma \mathcal{K}(x-y)\,\sigma(y)\,da(y) = \sum_{k=1}^{N_{panel}} \int_{\gamma_k} \mathcal{K}(x-y)\,\sigma(y)\,da(y) \\
&= \underbrace{\sum_{x\notin\mathcal{N}(\gamma_k)} \int_{\gamma_k} \mathcal{K}(x-y)\,\sigma(y)\,da(y)}_{\text{far-field}} + \underbrace{\sum_{x\in\mathcal{N}(\gamma_k)} \int_{\gamma_k} \mathcal{K}(x-y)\,\sigma(y)\,da(y)}_{\text{near interactions}}
\end{aligned}
$$

Boundary Quadratures

FLATIRON INSTITUTE
Center for Computational Mathematics

$$u(x) = \int_\Gamma \mathcal{K}(x-y)\,\sigma(y)\,da(y) = \sum_{k=1}^{N_{panel}} \int_{\gamma_k} \mathcal{K}(x-y)\,\sigma(y)\,da(y)$$

$$= \underbrace{\sum_{x \notin \mathcal{N}(\gamma_k)} \int_{\gamma_k} \mathcal{K}(x-y)\,\sigma(y)\,da(y)}_{\text{far-field}} + \underbrace{\sum_{x \in \mathcal{N}(\gamma_k)} \int_{\gamma_k} \mathcal{K}(x-y)\,\sigma(y)\,da(y)}_{\text{near interactions}}$$

Far field approximation:

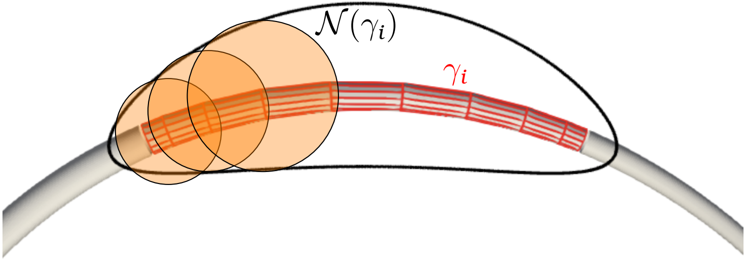
$$\int_{\gamma_k} \mathcal{K}(x-y)\,\sigma(y)\,da(y) \approx \sum_{i,j} \frac{2\pi w_i}{N_\theta} \mathcal{K}(x - y(s_i,\theta_j))\,\sigma(s_i,\theta_j)\,J(s_i,\theta_j)$$

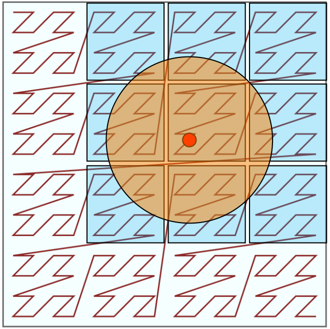
- Gauss-Legendre quadrature (s_i, w_i) of order q.
- periodic trapezoidal quadrature of order N_θ in θ.
- given a tolerance, define region $\mathcal{N}(\gamma_k)$, such that far-field is valid outside it.

Boundary Quadratures

$$u(x) = \int_\Gamma \mathcal{K}(x-y)\,\sigma(y)\,da(y) = \sum_{k=1}^{N_{panel}} \int_{\gamma_k} \mathcal{K}(x-y)\,\sigma(y)\,da(y)$$

$$= \underbrace{\sum_{x\notin\mathcal{N}(\gamma_k)} \int_{\gamma_k} \mathcal{K}(x-y)\,\sigma(y)\,da(y)}_{\text{far-field}} + \underbrace{\sum_{x\in\mathcal{N}(\gamma_k)} \int_{\gamma_k} \mathcal{K}(x-y)\,\sigma(y)\,da(y)}_{\text{near interactions}}$$





Boundary Quadratures

FLATIRON INSTITUTE
Center for Computational Mathematics

Near interactions: for $x \in \mathcal{N}(\gamma_k)$

$$\int_{\gamma_k} \mathcal{K}(x-y)\,\sigma(y)\,da(y) \;=\; \int_s \int_\theta \mathcal{K}(x-y(s,\theta))\,\sigma(s,\theta)\,J(s,\theta)\,d\theta\,ds$$

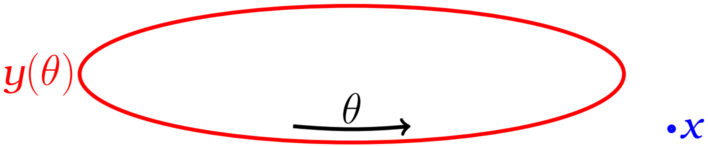
Inner integral:

- potential from a ring source (modal or toroidal Green's function).
- can be nearly singular as $s \longrightarrow s_0$.

Outer integral:

- singular if $x \in \gamma_k$ with logarithmic singularity at $s = s_0$.
- $1/s^\alpha$ decay as $|s - s_0| \longrightarrow \infty$

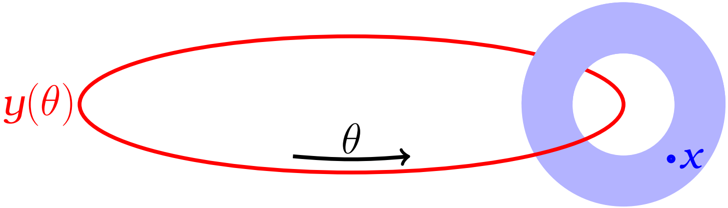
Fast Modal Green's Function Evaluation



$$\int_\theta \mathcal{K}(x - y(\theta))\, \sigma(\theta)\, d\theta$$

- Periodic trapezoidal rule becomes expensive as $x \longrightarrow y$.
- Analytic representation in special functions - Young, Hao, Martinsson JCP-2012
  - modal Green's functions -- method of choice for axisymmetric problems.

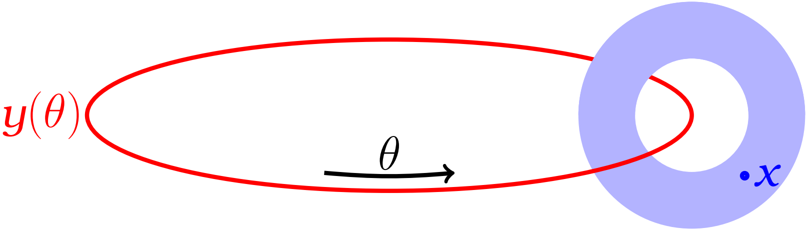
Fast Modal Green's Function Evaluation



$$\int_\theta \mathcal{K}(x - y(\theta))\, \sigma(\theta)\, d\theta$$

- Periodic trapezoidal rule becomes expensive as $x \longrightarrow y$.
- Analytic representation in special functions - Young, Hao, Martinsson JCP-2012
  - modal Green's functions -- method of choice for axisymmetric problems.
- Build special quadrature rules!
  - e.g. generalized Gaussian quadratures: Bremer, Gimbutas and Rokhlin - SISC 2010.

Fast Modal Green's Function Evaluation



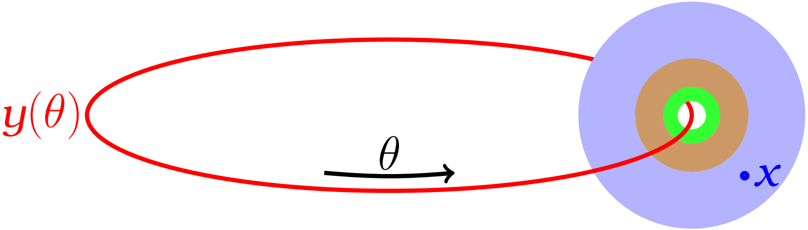
- Build special quadrature rule (w_i, θ_i) such that,

$$\int_\theta e^{-in\theta}\mathcal{K}(x - y(\theta))\, d\theta \approx \sum_i w_i e^{-in\theta_i}\mathcal{K}(x - y(\theta_i))$$

for all Fourier modes ($n \leq n_0$) and all targets x in the annulus.

Fast Modal Green's Function Evaluation

FLATIRON INSTITUTE
Center for Computational Mathematics



- Build special quadrature rule (w_i, θ_i) such that,

$$\int_\theta e^{-in\theta} \mathcal{K}(x - y(\theta))\, d\theta \approx \sum_i w_i e^{-in\theta_i} \mathcal{K}(x - y(\theta_i))$$

for all Fourier modes ($n \leq n_0$) and all targets x in the annulus.

- Different rule for each nested annular region (up to 10^{-6} from source).

$\sim$ 48 quadrature nodes for $n_0 = 8$ and 10-digits accuracy.
$\sim$ $26M$ modal Green's function evaluations/sec/core (Skylake 2.4GHz)

Quadratures for Outer Integral

$$\int_{\gamma_k} \mathcal{K}(x-y)\,\sigma(y)\,da(y) = \int_{\mathbf{s}} \left(\int_{\theta} \mathcal{K}(x-y(\mathbf{s},\theta))\,\sigma(\mathbf{s},\theta)\,J(\mathbf{s},\theta)\,d\theta \right) d\mathbf{s}$$

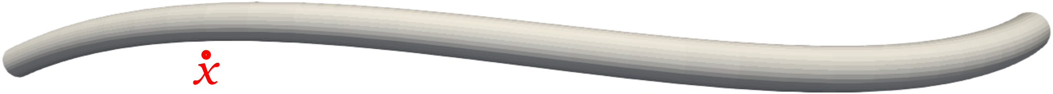
Quadratures for Outer Integral

$$\int_{\gamma_k} \mathcal{K}(x-y)\,\sigma(y)\,da(y) \;=\; \int_s \left(\sum_{n=0}^{N_\theta/2-1} \mathcal{K}_n(x-y(s))\,\widehat{\sigma_n} \right) ds$$

Quadratures for Outer Integral

$$\int_{\gamma_k} \mathcal{K}(x-y)\,\sigma(y)\,da(y) \;=\; \int_s \left(\sum_{n=0}^{N_\theta/2-1} \mathcal{K}_n(x-y(s))\,\widehat{\sigma_n} \right) ds$$

Near Interactions: x is off-surface or adjacent panel

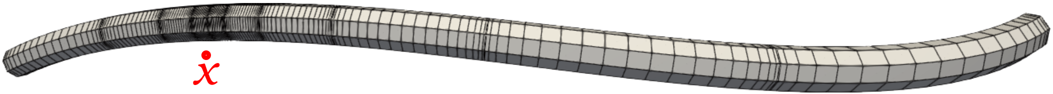


Quadratures for Outer Integral

FLATIRON INSTITUTE
Center for Computational Mathematics

$$\int_{\gamma_k} \mathcal{K}(x-y)\,\sigma(y)\,da(y) \;=\; \int_s \left(\sum_{n=0}^{N_\theta/2-1} \mathcal{K}_n(x-y(s))\,\widehat{\sigma_n} \right) ds$$

Near Interactions: x is off-surface or adjacent panel

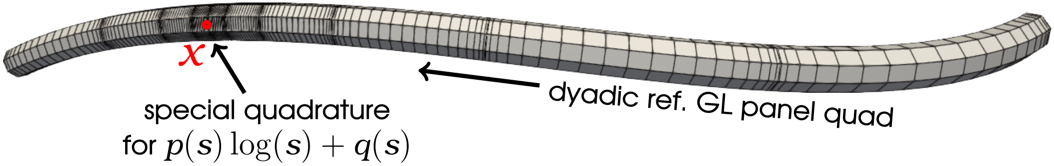


- panel (Gauss-Lengendre) quadrature with dyadic refienement.

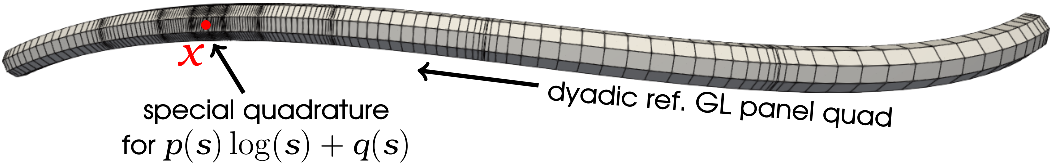
Quadratures for Outer Integral (singular case)

FLATIRON INSTITUTE
Center for Computational Mathematics

Quadratures for Outer Integral (singular case)



Quadratures for Outer Integral (singular case)

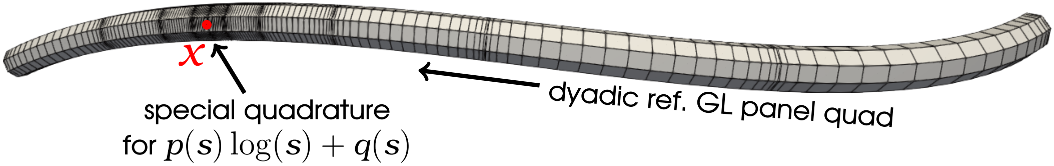


Special Quadrature Rules:

- replace composite panel quadratures with a single quadrature.
- integrand doesn't have closed form expression, but we can still generate quadrature rules!

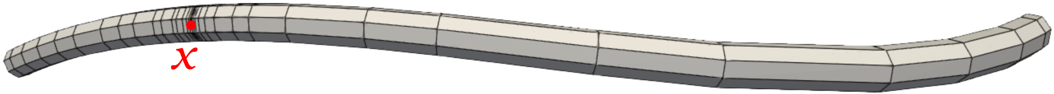
Quadratures for Outer Integral (singular case)

FLATIRON INSTITUTE
Center for Computational Mathematics



Special Quadrature Rules:

- replace composite panel quadratures with a single quadrature.
- integrand doesn't have closed form expression, but we can still generate quadrature rules!



- Separate rules for different aspect ratios (1 -- 10^4 in powers of 2)

Overall Algorithm

Discretization: piecewise polynomial $\times$ Fourier.

Far-field interactions: standard quadratures (GL $\times$ PTR) + FMM

Near interactions:

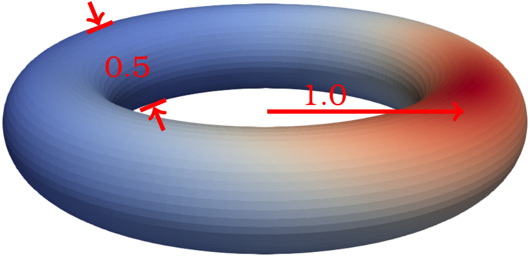
- special quadratures for modal Green's function and singular integral in s.
- dyadic refined Gauss-Legendre quadrature in s for non-singular case.
- build local correction matrix instead of computing on-the-fly.

Numerical Results - comparison with BIEST

Green's identity (Laplace):

$\Delta u = 0,$ then for $x \in \Gamma,$

$$u(x) = \frac{u(x)}{2} + \mathcal{S}[\partial_n u](x) - \mathcal{D}[u](x)$$



Boundary Integral Equation Solver for Taylor States (BIEST)*

- quadrature for general toroidal surfaces with uniform grid.
- partition-of-unity to separate singular part of boundary integral.
- polar coordinate transform for singular integral.

*JCP 2019 - Malhotra, Cerfon, Imbert-Gérard, O'Neil (https://github.com/dmalhotra/BIEST)

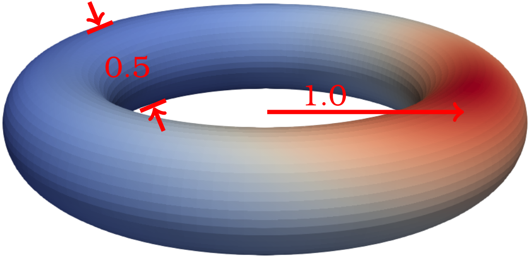
Numerical Results - comparison with BIEST

FLATIRON INSTITUTE
Center for Computational Mathematics

Green's identity (Laplace):

$\Delta u = 0,$ then for $x \in \Gamma,$

$$u(x) = \frac{u(x)}{2} + \mathcal{S}[\partial_n u](x) - \mathcal{D}[u](x)$$

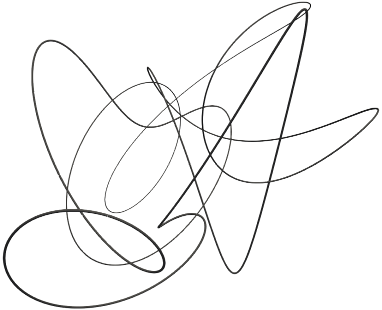


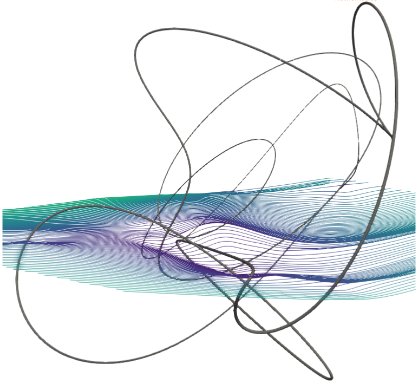
Slender-body Quadrature				BIEST*			
N	$\|\|e\|\|_\infty$	T_{setup}	T_{eval}	N	$\|\|e\|\|_\infty$	T_{setup}	T_{eval}
320	1.5e-04	0.032	0.0004	507	2.0e-03	0.1319	0.0017
720	3.5e-06	0.094	0.0013	1323	4.0e-06	1.4884	0.0042
1280	5.4e-09	0.228	0.0033	2523	4.3e-09	6.6825	0.0313
2000	2.5e-10	0.501	0.0079	4107	3.5e-10	15.4711	0.0862

* JCP 2019 - Malhotra, Cerfon, Imbert-Gérard, O'Neil (https://github.com/dmalhotra/BIEST)

Numerical Results - Stokes BVP

FLATIRON INSTITUTE
Center for Computational Mathematics





Exterior Stokes Dirichlet BVP:

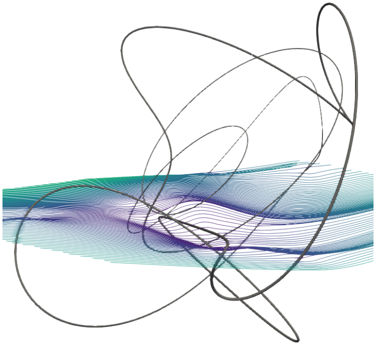
$$\Delta \boldsymbol{u} - \nabla p = 0,$$
$$\nabla \cdot \boldsymbol{u} = 0,$$
$$\boldsymbol{u}|_\Gamma = \boldsymbol{u}_0,$$
$$u(x) \to 0 \text{ as } |x| \to 0,$$

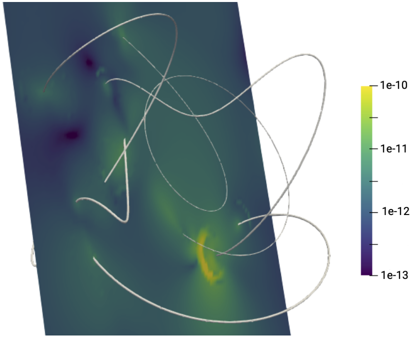
wire radius = 1.5e-3 to 4e-3

wire length = 16

BIE formulation: $(\mathcal{I}/2 + \mathcal{D} + \mathcal{S} \;/\; (2\varepsilon \log \varepsilon^{-1}))[\boldsymbol{\sigma}] = \boldsymbol{u}_0$

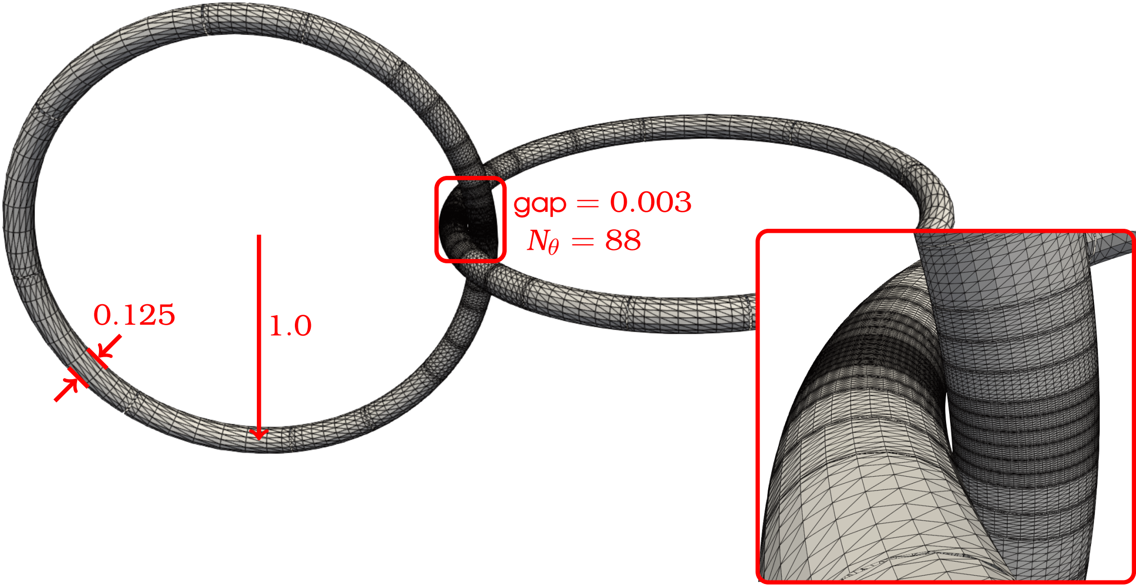
Numerical Results - Stokes BVP





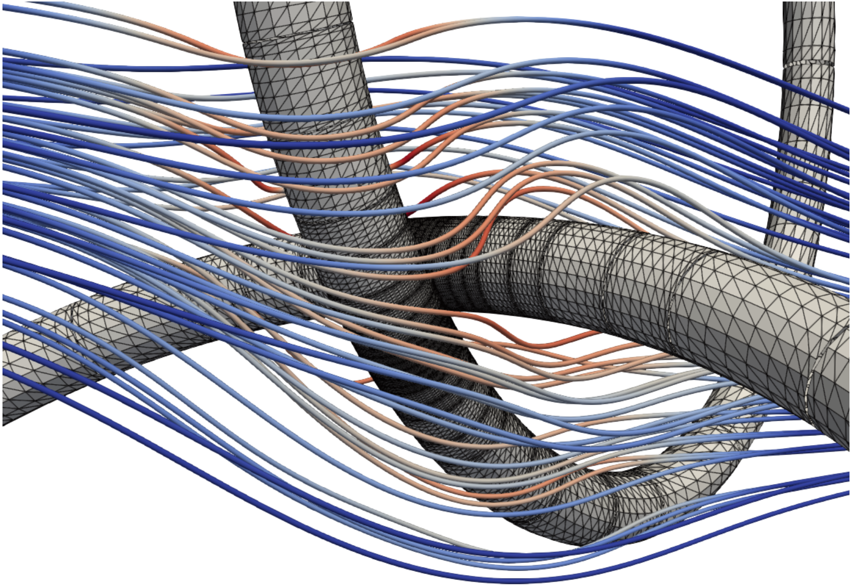
						1-core			40-cores	
N	N_{panel}	N_θ	ϵ_{GMRES}	N_{iter}	$\|\|e\|\|_\infty$	T_{setup}	(N/T_{setup})	T_{solve}	T_{setup}	T_{solve}
1.5e4	122	4	1e-03	10	1.9e-02	0.33	(4.4e4)	0.7	0.024	0.05
9.1e4	252	12	1e-05	21	1.7e-04	3.31	(2.7e4)	61.2	0.197	5.25
9.4e4	262	12	1e-07	33	4.1e-06	4.43	(2.1e4)	104.3	0.224	7.69
2.0e5	272	24	1e-09	43	1.4e-08	17.70	(1.1e4)	586.0	0.796	22.94
2.3e5	276	28	1e-11	54	4.1e-09	27.67	(8.4e3)	1034.2	1.229	38.85

Numerical Results - close-to-touching



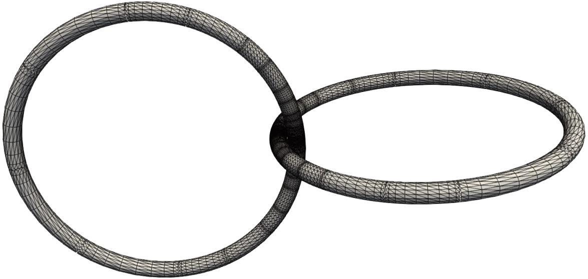
Numerical Results - close-to-touching

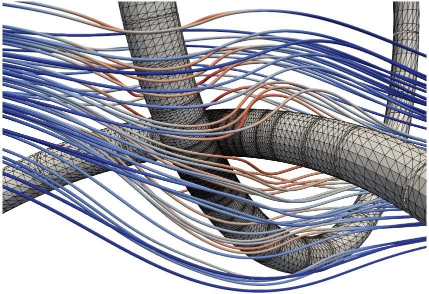
FLATIRON INSTITUTE
Center for Computational Mathematics



Numerical Results - close-to-touching

FLATIRON INSTITUTE
Center for Computational Mathematics





				1-core		40-cores	
N	ϵ_{GMRES}	N_{iter}	$\|\|e\|\|_\infty$	T_{setup} (N/T_{setup})	T_{solve}	T_{setup}	T_{solve}
6.5e4	1e-02	4	2.1e-02	8.1 (8.0e+3)	6.5	1.28	1.4
6.5e4	1e-05	24	2.4e-03	16.8 (3.8e+3)	42.9	2.50	7.7
6.5e4	1e-07	43	2.8e-06	23.5 (2.7e+3)	81.6	3.31	12.8
6.5e4	1e-10	59	5.4e-08	35.6 (1.8e+3)	122.9	4.06	19.2
6.5e4	1e-13	72	1.3e-10	49.9 (1.3e+3)	162.6	5.27	23.2

Mobility problem

Flatiron Institute
Center for Computational Mathematics

- n rigid bodies $\Omega = \sum_{i=1}^{n} \Omega_i$
  with velocities $\boldsymbol{V}(\boldsymbol{x}) = \boldsymbol{v}_i + \boldsymbol{\omega}_i \times (\boldsymbol{x} - \boldsymbol{x}_i^c)$,
  and given forces $\boldsymbol{F}_i$, torques $\boldsymbol{T}_i$ abount $\boldsymbol{x}_i^c$.

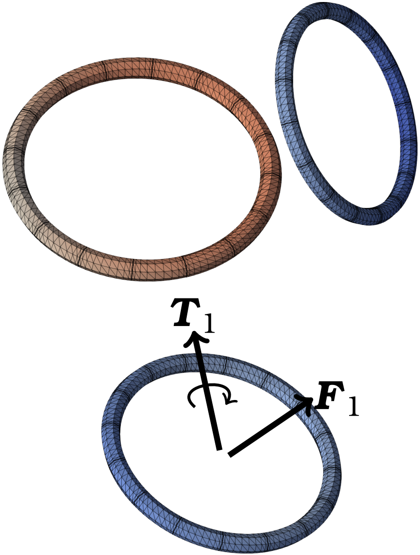
- Stokesian fluid in $\mathbb{R}^3 \setminus \Omega$

$$\Delta \boldsymbol{u} - \nabla p = 0, \quad \nabla \cdot \boldsymbol{u} = 0,$$

$$\boldsymbol{u} \to 0 \text{ as } \boldsymbol{x} \to \infty.$$

- Boundary conditions on $\partial\Omega$,

$$\boldsymbol{u} = \boldsymbol{V} + \boldsymbol{u}_s.$$



Mobility problem

- n rigid bodies $\Omega = \sum_{i=1}^{n} \Omega_i$

  with velocities $\boldsymbol{V}(\boldsymbol{x}) = \boldsymbol{v}_i + \boldsymbol{\omega}_i \times (\boldsymbol{x} - \boldsymbol{x}_i^c)$,

  and given forces $\boldsymbol{F}_i$, torques $\boldsymbol{T}_i$ abount $\boldsymbol{x}_i^c$.

- Stokesian fluid in $\mathbb{R}^3 \setminus \Omega$

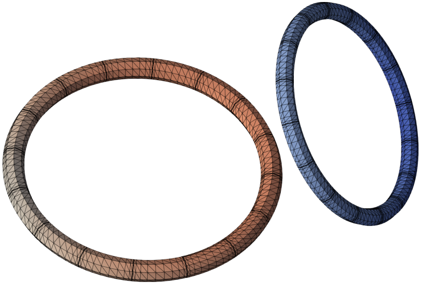
$$\Delta \boldsymbol{u} - \nabla p = 0, \quad \nabla \cdot \boldsymbol{u} = 0,$$

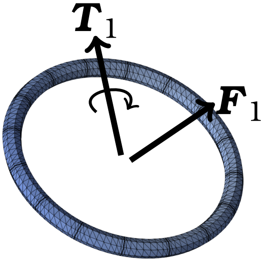
$$\boldsymbol{u} \to 0 \ \text{ as } \ \boldsymbol{x} \to \infty.$$

- Boundary conditions on $\partial\Omega$,

$$\boldsymbol{u} = \boldsymbol{V} + \boldsymbol{u}_s.$$

unknown: $\boldsymbol{V}(\boldsymbol{u}_i, \boldsymbol{\omega}_i)$





Mobility problem - double-layer formulation

Represent fluid velocity: $\boldsymbol{u} = \mathcal{S}[\boldsymbol{\nu}(\boldsymbol{F}_i, \boldsymbol{T}_i)] + \mathcal{D}[\boldsymbol{\sigma}]$

and rigid body velocity: $\boldsymbol{V} = -\sum_{i=1}^{6n} \mathfrak{v}_i \mathfrak{v}_i^T \boldsymbol{\sigma}$

Applying boundary conditions $(\boldsymbol{u} = \boldsymbol{V} + \boldsymbol{u}_s \text{ on } \partial\Omega)$,

$$(\mathcal{I}/2 + \mathcal{D})[\boldsymbol{\sigma}] + \sum_{i=1}^{6n} \mathfrak{v}_i \mathfrak{v}_i^T \boldsymbol{\sigma} = \boldsymbol{u}_s - \mathcal{S}[\boldsymbol{\nu}]$$

(Pozrikidis - Boundary Integral and Singularity Methods for Linearized Viscous Flow)

Mobility problem - double-layer formulation

Represent fluid velocity: $\boldsymbol{u} = \mathcal{S}[\boldsymbol{\nu}(\boldsymbol{F}_i, \boldsymbol{T}_i)] + \mathcal{D}[\boldsymbol{\sigma}]$

and rigid body velocity: $\boldsymbol{V} = -\sum_{i=1}^{6n} \mathfrak{v}_i \mathfrak{v}_i^T \boldsymbol{\sigma}$

Applying boundary conditions $(\boldsymbol{u} = \boldsymbol{V} + \boldsymbol{u}_s$ on $\partial\Omega)$,

$$(\mathcal{I}/2 + \mathcal{D})[\boldsymbol{\sigma}] + \sum_{i=1}^{6n} \mathfrak{v}_i \mathfrak{v}_i^T \boldsymbol{\sigma} = \boldsymbol{u}_s - \mathcal{S}[\boldsymbol{\nu}]$$

(Pozrikidis - Boundary Integral and Singularity Methods for Linearized Viscous Flow)

Second kind integral equation, should be well-conditioned.

What can possibly go wrong?

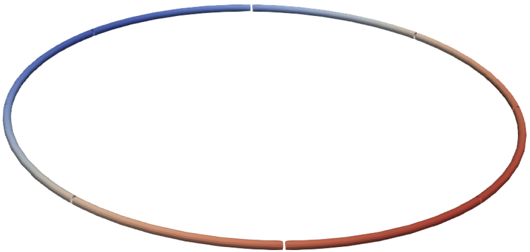
Conditioning of layer-potential operators

FLATIRON INSTITUTE
Center for Computational Mathematics

$$\kappa(\mathcal{S}) \sim 2.6e6$$

$$\kappa(\mathcal{I}/2 + \mathcal{D}) \sim 4.3e6$$

$$\kappa(\mathcal{I}/2 + \mathcal{D} + 16\mathcal{S}) \sim 80$$



- For infinite cylinder (Laplace case): $\kappa(\mathcal{I}/2 + \mathcal{D}) \sim \varepsilon^{-2} \log^{-1} \varepsilon^{-1}$
- Combined field operator well-conditioned: $\mathcal{I}/2 + \mathcal{D} + \mathcal{S} \;/\; (2\varepsilon \log \varepsilon^{-1})$

Mobility problem - combined field formulation

Represent fluid velocity: $\boldsymbol{u} = \mathcal{S}[\boldsymbol{\nu}(\boldsymbol{F}_i, \boldsymbol{T}_i)] + \mathcal{K}[\boldsymbol{\sigma}]$

and rigid body velocity: $\boldsymbol{V} = -\sum_{i=1}^{6n} \mathfrak{v}_i \mathfrak{v}_i^T \boldsymbol{\sigma}$

where, $\mathcal{K} = \mathcal{D} + \mathcal{S}/(2\varepsilon \log \varepsilon^{-1})$.

Mobility problem - combined field formulation

Represent fluid velocity: $\boldsymbol{u} = \mathcal{S}[\boldsymbol{\nu}(\boldsymbol{F}_i, \boldsymbol{T}_i)] + \mathcal{K}[\boldsymbol{\sigma} - \sum_{i=1}^{6n} \mathfrak{v}_i \mathfrak{v}_i^T \boldsymbol{\sigma}]$

and rigid body velocity: $\boldsymbol{V} = -\sum_{i=1}^{6n} \mathfrak{v}_i \mathfrak{v}_i^T \boldsymbol{\sigma}$

where, $\mathcal{K} = \mathcal{D} + \mathcal{S}/(2\varepsilon \log \varepsilon^{-1})$.

Mobility problem - combined field formulation

Represent fluid velocity: $\boldsymbol{u} = \mathcal{S}[\boldsymbol{\nu}(\boldsymbol{F}_i, \boldsymbol{T}_i)] + \mathcal{K}[\boldsymbol{\sigma} - \sum_{i=1}^{6n} \mathfrak{v}_i \mathfrak{v}_i^T \boldsymbol{\sigma}]$

and rigid body velocity: $\boldsymbol{V} = -\sum_{i=1}^{6n} \mathfrak{v}_i \mathfrak{v}_i^T \boldsymbol{\sigma}$

where, $\mathcal{K} = \mathcal{D} + \mathcal{S}/(2\varepsilon \log \varepsilon^{-1})$.

Applying boundary conditions,

$$(\mathcal{I}/2 + \mathcal{K})[\boldsymbol{\sigma} - \sum_{i=1}^{6n} \mathfrak{v}_i \mathfrak{v}_i^T \boldsymbol{\sigma}] + \sum_{i=1}^{6n} \mathfrak{v}_i \mathfrak{v}_i^T \boldsymbol{\sigma} = \boldsymbol{u}_s - \mathcal{S}[\boldsymbol{\nu}]$$

Second kind integral equation and well-conditioned!

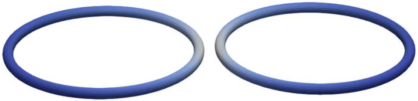
Numerical Results - Sedimentation Flow

FLATIRON INSTITUTE
Center for Computational Mathematics

Time-stepping: 5-th order adaptive SDC

8-digits accuracy in quadratures, GMRES solve, and time-stepping.

40 CPU cores

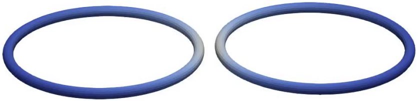


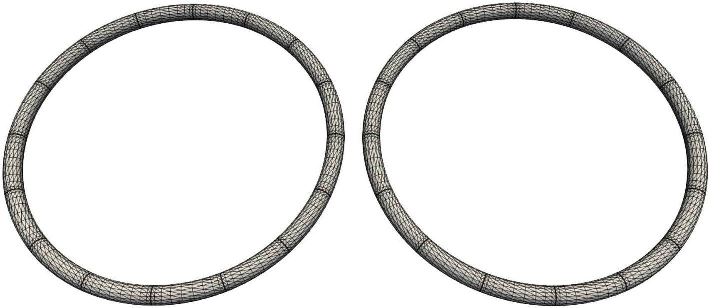
Numerical Results - Sedimentation Flow

Time-stepping: 5-th order adaptive SDC

8-digits accuracy in quadratures, GMRES solve, and time-stepping.

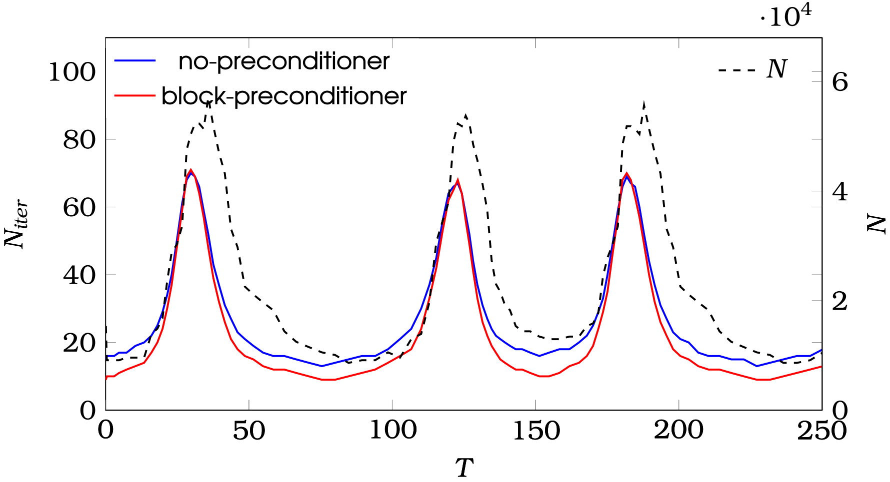
40 CPU cores





Numerical Results - Sedimentation Flow

FLATIRON INSTITUTE
Center for Computational Mathematics



Close-to-touching: smaller time-steps, more unknowns (N), high GMRES iteration count (block preconditioner doesn't help).

Numerical Results - Sedimentation Flow

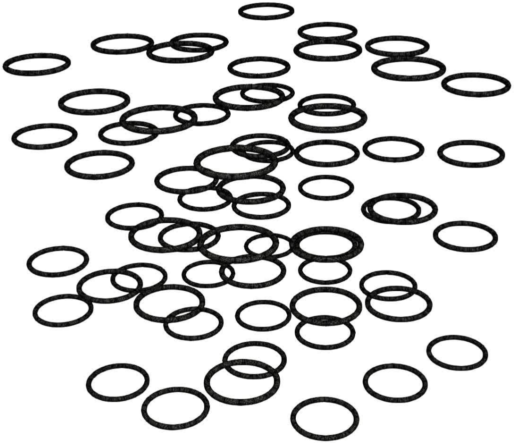
FLATIRON INSTITUTE
Center for Computational Mathematics

5-th order adaptive SDC

8-digits accuracy in quadratures, GMRES solve, and time-stepping.

0.5 million unknowns
64 rings.

160 CPU cores



Conclusions

FLATIRON INSTITUTE
Center for Computational Mathematics

- Convergent boundary integral formulation for slender bodies.
  - unlike SBT, boundary conditions are actually enforced to high accuracy.
- Special quadratures - efficient for aspect ratios as large as 10^5.
  - quadrature setup rates up to $20,000$ unknowns/s/core (comparable to FMM speeds).
- Stokes mobility problem - combined field BIE formulation.
  - well-conditioned formulation for slender-body geometries.
  - high-order time stepping (SDC), block-diagonal preconditioner.

Conclusions

- Convergent boundary integral formulation for slender bodies.
  - unlike SBT, boundary conditions are actually enforced to high accuracy.
- Special quadratures - efficient for aspect ratios as large as 10^5.
  - quadrature setup rates up to $20,000$ unknowns/s/core (comparable to FMM speeds).
- Stokes mobility problem - combined field BIE formulation.
  - well-conditioned formulation for slender-body geometries.
  - high-order time stepping (SDC), block-diagonal preconditioner.

Limitations and ongoing work:

- Open problems: collisions, better preconditions / fast direct solvers.
- Open ended and flexible fibers -- applications in biological fluids.